5

On The Nature of Competence: Principles for Understanding in a Domain©

Rochel Gelman
University of Pennsylvania

James G. Greeno
University of California, Berkeley

It is now commonplace that a theory of instruction requires the three significant components identified by Glaser (e.g., 1976): a theory of the knowledge that we want students to acquire, a theory of the initial state of the learner, and a theory of the process of transition between the initial state and the desired state of knowledge to be achieved in instructional settings. In this chapter we review and integrate the research that we, with our students and associates, have done to characterize the initial knowledge states that learners bring to mathematics instruction. Previous work shows that children have significant implicit understanding of counting, number, and sets before they receive instruction (e.g., Gelman & Meck, 1986; Hudson, 1983; Resnick, 1986). Our models characterize this understanding in the form of schemata that can be used for generating plans for both performance of tasks and representations of information in texts and situations presented as problems.

Our studies and analyses of early mathematical competence emphasize nor known in mathematical terms, what do we mean? The idea is that a skeletal set of principles is available to support the kind of selective attention and learning in the domain characterized by these principles. Initial representations serve as enabling devices. Learning would be exceedingly hard, if not impossible, were there no such conceptual skeletons supporting the growth and development of body parts (principles) and the accrual of the relevant body of knowledge. In the end, surely as a function of everyday experiences and "healthy" instruction, both the body and the skeleton will develop together. Eventually, the child will come to understand counting (and other principles of arithmetic) *at many levels*, enough that even the claim that she *re-presents* number and other mathematical concepts in the stands-for, symbolic sense is correct. Indeed, a major goal of mathematics instruction is to make students fluent speakers of the language of mathematics. Later we develop the theme that the road to this goal involves the acquisition of layers of representation. Successive layers feed off ones already available or acquired.

Granting the young principles does not mean we assume they know everything about these and related ones. There will be much for them to learn; but they are given a push in the right direction.

In this chapter we develop and extend our position that young children have implicit understanding of some counting principles (e.g., Gelman & Gallistel, 1978; Gelman & Meck, 1986; Gelman, Meck, & Merkin, 1986; Greeno, Riley, & Gelman, 1984), by using the Greeno et al. (1984) proposal that competent plans for counting reflect the ability of *procedural competence* to generate plans that honor the constraints dictated by the domain of knowledge as well as the constraints dictated by the task and setting. We assume that the ability to use these kinds of constraints depends on the nature of what we have called *conceptual competence* and *utilization competence*, respectively. Our proposal that competent plans for counting reflect multiple components of competence has implications regarding the interpretation of variable performance and an account of how early numerical understandings might develop.

First, we cover background issues and a summary of our previous work. Next, we extend the Greeno et al. analysis, especially with regard to the question of how to characterize competence involved in linguistic or symbolic interpretations of situations in the domain. After that, we take up topics related to the interpretation of variability in performance, that is, the competence–performance distinction. The end of the chapter includes suggestions about how competence models like ours contribute to a theory of active learning. We present and discuss some suggestions on how to develop links between our analyses of the initial states of learners and the theory of transition that is needed if we are to have a theory of instruction.

Because we believe our approach to modeling understanding of counting and arithmetic can be applied to other bodies of knowledge that are organized in terms of a set of principles and the entities to which those principles apply, we make an effort throughout the chapter to use examples from other domains to illustrate points.

WHY POSTULATE PRINCIPLES?

Fundamentally, we postulate principles to provide a basis for defining knowledge in a domain. In the case of counting, implicit knowledge of the stable-order principle underlies our understanding that one need not use the conventional count words of a language to count. As long as some ordered set of tags is used, be it comprised of verbal elements or not, and as long as one uses these in a way that honors the one–one principle, we can say they are using counting tags. In other words, the tags are counting tags if their use honors the constraints of the counting principles. Given this, if we grant people implicit knowledge we provide them a tool for identifying the many acceptable sets of tags used throughout the world. If the structure and use of a set fails to meet the constraints of the principles, as is the case for the alphabet in English, but not Hebrew, then the list is *not* a count list.

When counting a set, one does not have to point to each item, start at the beginning of a row, arrange items in a row, count anything in particular, and so forth. There are multiple ways to keep track of the number of items used in a count-on strategy (Fuson, 1982), just as there are multiple count strategies that can be used to solve a variety of arithmetic problems. All these count strategies are united by their function: They solve the problem of determining the cardinality of a set. They do not determine the cause of an object's acceleration or one's memory for places, or even the area covered by a particular object.

How are we able to determine that these disparate events are in the equivalence class of procedures for counting? What gives us license to say that a variety of distinct behaviors are all instances of the class of counting behaviors? What leads us to say that attributing cause and choosing a counting direction are not in the same class? Our answer is knowledge of the counting principles. Our representation of these principles serves to define what counting is and underlies our ability to identify members of the equivalence class of counting behaviors. Just as principles of syntax help define the class of acceptable utterances in a language, so do principles of counting help define the class of acceptable instances of counting.

The fact that principles define the equivalence class of counting behaviors is related to another characteristic they possess. They serve as the basis from which novel instances can be generated; they yield constraints that a stance, surface characteristics, weight, etc. of objects. For such considerations influence the choice of device for moving that object, decisions as to how fast the object will move, and so on.

There are two further reasons to maintain that understanding in a domain involves the implicit use of domain's principles. The first is that we get an enriched analysis of the competence–performance distinction. We see later that this happens when we separate the notion of competence within a domain from competence for assessing a setting within which to display the domain-specific competence as well as competence for generating a plan of action that honors the constraints of both the domain's and the settings's requirements. Doing this leads to the realization that the ability to generate a competent plan of action depends on all three of these abilities. It also highlights the fact that once one has a competent plan of action, the plan still needs to be executed. Failure to execute the plan could be due to yet further variables, especially those that are traditionally associated with performance problems (e.g., Flavell, 1970). In sum, two issues need to be considered in treatments of competence. The first concerns the need to distinguish between competence that is required for the domain and a competent plan of action. A competent plan is required for correct performance in a specific task setting, so even if a child has the competence corresponding to all the counting principles, if she lacks some of the interpretative competence (discussed later) or planning competence needed to generate the requisite counting procedures for a given setting, she will not succeed. Such a child would generate incorrect plans for counting, in spite of the fact that she has counting-principle conceptual competence. Additionally, even if a child does generate a competent plan, skills required for execution might surpass her performance abilities. In this case, a different source of difficulty contributes to the variability of performance levels.

Second, our approach provides some clues as to how to account for self-initiated and self-guided learning of further knowledge within a domain. Because principles underlie selective attention to domain-relevant inputs, it follows that the mental structures involved serve to interpret environments. The same structures can provide at least a match–no match test of whether the environment and behavior does meet the constraints of the domain. Hence, the conclusion that our position allows for the development of a theory of self-initiated monitoring of the adequacy of stimuli and responses to same. If we add the assumption that those environments that set up structural isomorphs between the definition of domain-relevant stimuli and the ones presented in the environment are especially salient, we start to have a handle on the kind of mechanism that might support the learning of further facts and/or principles of the domain in question. Later we expand on these points.

What Counts as Evidence?: Clues From Language Acquisition

We say children have understanding of some of the principles that govern a domain of knowledge, even when they cannot state the principles, if their behavior reveals implicit or tacit use and understanding of the principles. Our attribution of principled understanding assumes that correct performances are not due simply to some ability to produce a chain of responses memorized by rote in the context of the task in question. Thus, we need evidence to rule out the rote-learning alternative. We turn to the study of language for an analogous case that provides such evidence.

First, when intending to talk, speakers produce sound sequences that are identifiable as utterances as opposed to melodies or other kinds of patterned sounds. This suggests the use of a speech planner constrained by the principles of phonetics, morphology, and syntax. Second, even though speakers sometimes produce ill-formed utterances, their utterances are almost always novel. To be sure, some phrases are used repeatedly, like "How are you?" and "Excuse me."; but, by and large, even beginning language users produce novel utterances that are recognized as acceptable instances of the language. This ability to generate novel utterances provides a major source of evidence that the speaker–hearer makes use of principles. For otherwise, it is hard to explain the ability to use and combine speech relevant units, be they phonemes, morphemes, words, or phrases.

Another indication that implicit principles of understanding guide the language user comes from error-detection data. "What did Frank buy and?" is not an acceptable sentence to speakers of English, even though they may have never encountered the entire string or the acceptable string within which it is embedded. Our ability to sort strings of words that are acceptable from those that are not provides yet another line of support for the conclusion that we have implicit, principled understanding of the rules of syntax that guide and constrain the selection and combination of words. For strings generated in accord with these constraints are possible instances of an utterance; those that violate these constraints are not. In the present example the *coordinate constraint* (Ross, 1967) is violated. When the string is altered to honor the constraint, as in "What did Frank buy and keep?," our implicit understanding of the rule that embodies the constraint leads us to accept the sentence. Implicit knowledge of such principles makes it possible for us to identify members of the equivalence class of responses that are judged acceptable.

To summarize, people are said to have principled knowledge of language because, in addition to being able to produce acceptable utterances, they produce novel utterances and can discriminate between novel accept-order, and cardinality—and two conditions-of-application principles—item-irrelevance and order-irrelevance of objects. The three how-to principles capture the facts that counting requires tagging uniquely each item in a set of objects with a stably ordered list of tags, and the last tag in the list can be used to represent the number of objects in the set. The other two principles reflect the mathematically relevant consequences of the how-to constraints, namely, that any collection of discrete items can be counted, and that the order in which these are counted is irrelevant as long as the how-to principles are honored.

Two sets of schemata are included in the Greeno et al. (1984) characterization of conceptual competence. Schemata of *domain-specific* competence represent the constraints that the principles place on the generation of actions in a given setting. For example, schemata are included that specify that, in a count, tags must be applied to the items being tagged so that one and only one tag is applied to each item to be counted; a schema called KEEP-EQUAL INCREASE expresses this constraint.

The second part of conceptual competence has schemata such as PICK-UP, which characterize classes of behaviors that help satisfy the principles as specified. For example, the PICK-UP schema provides a means for satisfying some of the requirements specified in the domain-specific part of conceptual competence because it is one way to transfer items from the to-be-counted to the already-counted set as count tags. Such accomplishment-oriented schemata could be used for other purposes; hence, the fact that they serve this function in counting is surely learned. We refer to these schemata as *domain-linked* competence because they are recruited to satisfy requirements that arise from domain-specific competence but are not unique to counting. (See the next section for further discussion of domain linked vs. domain specific.)

Utilization propositions provide connections between features of the problem setting and goals of the planner. These propositions allow the planner to infer that some goals or subgoals can be achieved using features of the task setting. The planner includes a theorem prover that attempts to show that requisite conditions of actions can be satisfied in the setting. An example involves the arrangement of objects to be counted. If the objects are in a straight line, they can be counted by starting at one end and moving across the line. The position in the line at any time provides the information needed in remembering which objects have and have not been counted. Knowledge that a straight line of objects can be used to partition the set is a utilization proposition.

The planner that uses action schemata and utilization propositions to achieve goals contains *procedural competence*. The planner uses the knowledge sources of conceptual competence and utilizational propositions as well as general rules about the way to deal with action schemata. The

planner has knowledge that recognizes a goal and searches for an action schema with a consequence that matches the goal. The planner also identifies requirements of schemata as subgoals that need to be the subject of further planning.

The claim that competence, as characterized, is sufficient to determine successful performance is buttressed by an analysis by Smith, Greeno, and Vitolo (1985), who implemented a program that constructs plans for counting using the information structures and procedures assumed in the competence analysis. Their program implements a planning procedure and generates plans for counting in various settings when given components of knowledge about actions and features of task settings corresponding to the action schemata and utilization propositions of Greeno et al.'s (1984) analysis. Smith et al. (1985) also showed that the system can generate novel plans for counting when new constraints are imposed on the procedure, such as requiring that a designated object be associated with a designated numeral. It does so for the cases given, without adding schemata to the description of conceptual competence.

Demarcating Domain-Specific From Domain-Linked Competence. As discussed earlier, principle-driven models of understanding can be used to define domains as well as the nature of the equivalence class of behaviors in that domain. Based on a specified class of tasks that are judged to be in the domain of counting, we can determine those components of the competence that are necessary and sufficient to derive plans for correct performance. Judgments are required to determine whether a task involves counting and whether a specific performance of the task is correct, but these seem no more problematic than in the case of language where judgments are needed to determine whether strings of words are or are not grammatical.

As an example, we judge that counting includes settings where objects are arranged in a straight line and are tagged in a sequence from one end to another with corresponding increments in the numerals. Counting also includes settings where objects are moved into a designated location with an increment of the numeral string for each object. A third case that we need not judge as counting is construction of a set of tokens that is numerically equal to a set of objects—for example, creating a set of coins that is equal to a given set of books by placing a token in a bag corresponding to each book. Although, the last case involves a principle of one–one correspondence, it does not involve the principles of stable ordering and cardinality.

We say that a schema is part of domain-specific competence if it is included in the derivations of plans for all the tasks in the domain and if its removal allows derivation of plans that give incorrect performance. We say object as it is counted. This does not mean that a different understanding of counting principles is elicited by objects that are moved from that elicited by objects that are in a line. The required understanding of counting is the same, but different resources are available for achieving one of the requirements implied by the counting principles. Different utilization principles apply to the different situations, enabling the planner to choose appropriate counting-linked actions.

As another example, if items to be counted are arranged in a circle rather than a line, the plan for counting requires marking the first item counted. This action is not required when the objects are in a straight line, although the procedure of pointing to the objects in turn is used by many children in this setting, as it is when there is a straight line of objects. The circle display introduces a setting-relevant problem, one that requires an action of explicitly marking the beginning and ending of the set. Unless the starting point is remembered correctly, one–one correspondence could be violated by stopping too soon or continuing too long with the counting procedure. Utilization propositions apply in the straight-line case to infer that counting is complete when the end of the line is reached, and for the circle case, to recognize that a test for completion is needed.

Some counting-linked actions can come to be used as if they are instances of schemata that are counting specific. For example, children learn a procedure for keeping track of counted objects in a line by pointing to them as they count. The schemata for this are the same as for keeping track of which numeral was used last and are probably assimilated to counting-specific competence, at least at some point in development, because young children appear to need them for all counting tasks involving 3-dimensional and 2-dimensional rows of objects. However, the actions of moving along a line of objects by pointing to them in turn are counting-linked actions only when objects are arranged in a straight line. Hence, the calling up of this knowledge is dependent on the setting and interpretative competence in that setting. So strictly speaking, such actions are not counting specific. Still, we believe they can function this way at times because they are isomorphic to a counting-specific schema.

We note an analogy between the ability to demarcate knowledge that is specific to a domain of procedures and the theoretical criterion that Chomsky (1965) called ***descriptive adequacy***. Chomsky proposed that a grammar should account for relations of similarity among sentences of a language, for example, for the classes of sentences that are judged to be paraphrases of each other. The ability of the counting competence analysis to account for intuitions that a class of procedures should all be judged to be counting examples is a form of descriptive adequacy that has special utility in clarifying the boundaries of domain-specific and domain-general understanding.

Our analysis also clarifies the distinction between competence and performance. In fact, it reveals that there are two distinctions involved. First, there is the distinction between domain-specific competence and competence for generating competent plans of action for a specific task setting. A child might have the competence corresponding to all the counting principles but lack some of the utilization propositions or action schemata needed to generate competent plans that make possible correct counting procedures in a setting. Such a child would generate faulty plans for counting, in spite of having all the domain-specific competence, because he or she would lack knowledge that is outside the domain of counting that is needed for successful planning in the specific task setting.

A second distinction is between the generation of a competent plan and the successful execution of a plan. Successful execution of counting requires holding information in memory, attending to the objects to be counted, and other information-processing requirements. The scheme shown in Fig. 5.1 accounts for the construction of a plan of action, but not for the plan's execution. A child could generate a correct plan for counting but then forget where he or she is in the process of counting, or forget a component of the plan, or skip an object because he or she does not notice it is there. Such failures of execution would count as incorrect performance in a narrower sense than failures of competence outside the domain of counting principles. We have not considered requirements of execution in our earlier discussions, but we give some attention to them here and in the next sections.

EXTENSIONS OF THE ANALYSIS OF COMPETENCE

One goal of this chapter is to develop and extend our notions about the components of competence. As we worked with the Greeno et al. model, we recognized that failure due to the absence of knowledge of a principle should be distinguished from failure due to the lack of the domain-relevant knowledge, for example, that a partitioning schema and the English string of words, "one, two, three, four, five," and so forth can serve counting. Whereas the principles define the domain, the latter do not. Earlier, we discussed why one need not use the count words to count. A partitioning schema can serve many functions, including classification and any other procedure that is to be applied to all members of a set. Still, even though other ordered lists can serve as tags, the one that is chosen becomes numerically meaningful because it is used in a way that satisfies the constraints of the counting principles. Similarly, methods of partitioning take on counting-specific meaning because they serve to honor the one–one constraints. These considerations have encouraged us to subcategorize conceptual

An Extended Model of Interpretative Competence. The idea that children's interpretations of task requirements benefit from structures of competence is familiar and compelling. For, without them, there would be no basis from which to select and assess the domain-relevant information in a given task, be it verbal or otherwise. In Piagetian terms, there would be no possibility of assimilating the environment correctly because there would be no appropriate structure to project onto (interpret) the environment. Whereas correct understanding of some components of the task setting is intricately linked to conceptual competence, understanding of other features is not. The vocabulary of the domain and the arrangement of items in a setting are variables that are so linked. They are also the kind of variables featured in the Greeno et al. analysis of utilization competence. We can now call the related propositions counting-linked utilization propositions.

Gelman et al.'s (1986) analysis of why children failed Baroody's (1984) test of the order invariance principle did not focus on such domain-specific variables; instead it highlighted the abilities to interpret social and conversational contexts. Limits on these abilities can lead one to a misunderstanding of the intended meaning of an experimenter's question (Donaldson, 1978). For this reason, we see the need to extend our notions of interpretation requirements to include the ability of the child to converge on the goal the experimenter has in mind. To do this, the child must have the ability to assess correctly both the setting and the conversational frame.

Understanding the different classes of social settings, syntax, and rules of conversation are not dependent on conceptual competence for counting, although they do contribute to communicative competence (Hymes, 1967); and therefore, what we call *interpretative competence*. If children set a counting goal that takes account of the counting-relevant variables but is not the one the experimenter had in mind, they will not succeed. An interpretative error of this kind will generate a failure on a particular task, even if the child has the requisite conceptual competence of the domain, a correct interpretation of the domain-relevant features of a task, and adequate procedural competence.

To address these questions of interpretation, we have added to our analysis the components of competence in Fig. 5.1 called *social and conversational schemata* and schematic knowledge of *word meanings*. Social and conversational schemata feed children's understandings of the task, including the questions they try to answer. The general social setting creates a context for the interaction between the child and the interviewer. The child's understanding of this setting is constructed using social schemata that he or she has about interacting with adults. This understanding includes communication goals that the child understands as operating in the conversational interaction of the interview. The child's understanding also includes his or her interpretation of speech acts, based on schemata of

conversational patterns such as being asked to give information that the questioner already knows.

In the context of the communicative interaction and the task setting, the interviewer presents domain-relevant information and asks questions about this information. Meanings of terms that refer to sets, numbers, and other domain-specific terms constitute further components of competence in the domain of counting. The process of interpreting the information presented linguistically includes a grammar that constructs meanings of phrases and sentences out of the meanings of words. Interpretation also includes setting goals for actions that can produce answers for questions.

An Example. To illustrate our motivation for modifying our analysis of utilization competence, we turn to Baroody's (1984) study of the order-invariance counting principle. Baroody challenged our conclusion that preschool children have implicit knowledge of the order-irrelevance principle. He suggested instead that they may have mastered a more restricted order-irrelevance rule, one that allows them to count the same row from either end and yet not appreciate that the cardinality of a set is preserved across any and all the different count orders. To assess this possibility, Baroody asked 5- and 6-year-old children to count a row of eight objects from left to right. This done, the experimenter covered the display and pointed to the place of the right-most item and said, "Could you make this the number one? (All children agreed they could.) We got eight (or whatever was the child's value) counting this way. What do you think we would get counting the other way?" The vast majority of the 5-year-olds responded with some value other than eight.

Gelman et al. (1986) suggested that the younger children in the Baroody experiment misinterpreted the experimenter's second question about cardinality. Perhaps the children did not treat the second question as a straightforward request for information about their knowledge of the conditions under which cardinal value is preserved and instead took it as a challenge to the correctness of their first answer. In the context of the present discussion, we can say their and the experimenter's interpretation of the goal differed. The experimenter intended they report the same cardinal value, if they believed the set size was the same; however, the younger children assumed they were supposed to find a goal that would allow them to give a different answer.

Support for this proposal comes from the demonstration that an altered question format led the younger children in Gelman et al. to do as well as an older control group. Instead of referring to their first count, Gelman et al. asked, "Could you count starting with the 'five'?" "What will you get?" or "How many will there be?" Further, children who participated in Gelman's Baroody Control condition and changed their answers did so in a concepts and principles is being developed by Greeno, Walter Johnson, and Adele Goldberg. (A preliminary report was in Greeno and Johnson, 1984.)

Effects of Wording on Children's Performance. At one level it is obvious that children's ability to succeed in tasks certainly depends on their understanding of language that is used to describe the tasks and to ask questions. If questions were asked in English to a child who only understood Spanish, no one would be surprised if the child failed. Several research findings have shown, however, that effects of language understanding can be considerably more subtle than total lack of comprehension.

Markman and her associates have introduced an important set of phenomena involving influences of wording on interpretation (Markman, 1979; Markman & Seibert, 1976). Markman studied the effect of referring to sets of objects with collection terms, such as "forest" or "football team," rather than class terms, such as "trees" or "football players." Children were more successful on tasks involving class inclusion and other quantitative relations, if the conversation and questions included references to collections.

One of the tasks studied by Markman was class inclusion (Markman & Seibert, 1976). When children mistakenly say there are more of a subset than of the set that contains it, they are often said to lack understanding of the hierarchical structure of subset relations (e.g., Inhelder & Piaget, 1959/1964). This conclusion may not be warranted. Children tested with collection terms (e.g., "Here is a bunch of grapes; there are green grapes and there are purple grapes, and this is the bunch. Who would have more to eat, someone who ate the green grapes or someone who ate the bunch?") answered correctly more often than children tested with descriptions and questions stated entirely in class terms (e.g., "Here are some grapes, there are green grapes and there are purple grapes; who would have more to eat, someone who ate the green grapes or someone who ate the grapes?").

Tasks used in Markman's experiments included conservation of number, questions about the cardinalities of sets following counting, and other situations in which she found that collection terms facilitated performance. Markman's data fit with our hypothesis that interpretative competence plays a role in representation of the task. In subsequent studies, Fuson, Pergament, and Lyons (1985) and Fuson (1986) have failed to replicate some of Markman's findings, so there may be other factors in addition to the wording of descriptions and questions that influence the phenomena. However, as we show later in this section, the likely role of collection terms is to encourage representations involving references to sets, rather than to require them, which would be consistent with the phenomena occurring in some contexts and not in others. Our analysis provides a reason for expecting that the collection-term effect might be more robust in class-inclusion tasks than in other quantitative judgments, and this is consistent with Fuson's (1986) success in replicating that finding.

Another effect of linguistic factors on children's success was demonstrated by Hudson (1983). Hudson was interested in children's difficulty with question about differences between cardinalities of sets. If children are shown pictures of two sets and are asked "How many more?"—for example, show five birds and two worms and ask "How many more birds are there than worms?" children often answer with the number of the larger set rather than the difference—that is, "five," rather than "three," in the case of five birds and three worms. This has often been interpreted as indicating that the children lack understanding of a concept of one-to-one correspondence. However, a different question leads to quite different results. For example, if the interviewer says, "The birds are going to race over and each one is going to try to get a worm. How many of the birds won't get a worm?" significantly more children answer correctly. It seems that children are able to compare the sets by forming sets with one-to-one correspondence and counting the remainder, when enough linguistic cues are provided.

Theoretical Framework. The analysis that Greeno, Johnson, and Goldberg (in progress) are developing provides hypotheses about competence for understanding descriptions and questions about numbers and sets of objects.

Fig. 5.2 shows an expansion of some of the components of interpretive competence shown in Fig. 5.1. There are three main components of the new analysis: (a) Analyses of propositional representations and intensions provide hypotheses about understanding of language about numbers and sets; (b) analyses of actions performed to make inferences and answer questions include hypotheses about competence for decisions made in determining sets to be counted and performing counting activities; (c) analyses of actions performed to construct models based on propositions that provide concrete representations that can be used to facilitate reasoning include hypotheses about understanding relations between predicates in propositions and sets that can be constructed.

Propositional representations and intensions are analyzed using Montague grammar (Dowty, Wall, & Peters, 1981), a system that provides formal methods for deriving meanings of phrases and sentences from assumptions about meanings of individual words.[1] Different levels of under-

[1]We recognize that many concepts cannot be defined with reference to intensional and extensional components. However, number concepts are a different matter and set theoretic models of them are common. Hence, our choice of a Montague Grammar to model the understanding of mathematical terms seems reasonable.

ence to sets is included in the meanings of propositions that have numerals and other quantifiers, such as "some." These meanings reflect an understanding of numbers as the cardinalities of sets; in effect, when a child hears a phrase such as "three marbles," he or she understands that there is a set of marbles and that "three" denotes the cardinality of that set. We therefore say that the competence for understanding propositions that include reference to sets includes understanding of a principle that we call *linguistic cardinality*.

At a third level, numerals also denote the numerical differences between sets. At this level, the meaning of a sentence such as "Kay has two more marbles than Jay" includes reference to the set of Kay's marbles, the set of Jay's marbles, and a third entity, the numerical difference between the two sets. In this usage, numbers are properties of a relation between sets, and a child's concept of number must therefore be more complex than is needed for understanding numbers only as cardinalities of individual sets. We say that the competence for understanding propositions that include reference to set differences includes a principle of *linguistic numerical difference*.

Linguistic Cardinality for Representing Single Sets. To illustrate the distinction between the first two levels, consider the sentence "Jay has three bowls." First, we show a propositional representation at the first level, where linguistic cardinality is not assumed.

1 "Jay has three bowls" $\Rightarrow \exists x \exists y \exists z[\text{bowl}(x) \wedge \text{bowl}(y) \wedge \text{bowl}(z) \wedge \text{has}(\text{Jay},x) \wedge \text{has}(\text{Jay},y) \wedge \text{has}(\text{Jay},z) \wedge x \neq y \wedge x \neq z \wedge y \neq z]$.

This logical notation is read, "There is an *x*, there is a *y*, and there is a *z*, such that *x* is a bowl, *y* is a bowl, *z* is a bowl. Jay has *x*, Jay has *y*, Jay has *z*. *X* and *y* are not the same, *x* and *z* are not the same, and *y* and *z* are not the same (i.e., *x*, *y*, and *z* are distinct objects). The important feature of the proposition for us is that it refers separately to the three objects that are Jay's bowls, rather than referring to the set of bowls.

In contrast, a propositional representation with linguistic cardinality is:

2 "Jay has three bowls" $\Rightarrow \exists X[3(X) \wedge \forall x(x \in X \rightarrow \text{bowl}(x)) \wedge \forall x(x \in X \rightarrow \text{has}(\text{Jay},x))]$.

By convention, we use capital letters for variables that have sets of objects as values and lower case letters for variables that have individual objects as values. The second logical form is read, "There is a set *X*, such that *X*'s cardinality is 3, and for all *x*, if *x* is a member of *X*, then *x* is a bowl (i.e., all the members of *X* are bowls), and for all *x*, if *x* is a member of *X*, then Jay has *x* (i.e., Jay has all the members of *X*).

Short

The derivation of these representations is based on assumptions about the meanings and semantic categories of the individual words in the sentences. The difference between 1. and 2. depends on the meaning of the word *three*. In the current analysis of Greeno et al., numeral terms are in the same semantic category as determiners, such as *a* and *the*. The meaning of "Jay has a bowl" includes reference to a bowl—that is, for "Jay has a bowl" to be true, there must be an object that is a bowl that Jay has. The meaning of "Jay has three bowls" is analogous. For "Jay has three bowls" to be true there has to be something that "three bowls" denotes and that Jay has. In *1*, the something is three individual bowls, and the meaning of three includes references to three individual objects. In *2*, the something is a set of objects with cardinality *3*, and the meaning of three includes reference to that set.

Propositions are descriptions of situations, and the intensions of terms in a propositional representation are mappings from situation indices to the objects that terms denote. Intensions exist for the terms in a proposition for situations, where the proposition is true. For example, suppose that there is a world containing five ceramic bowls, a, b, c, d, and e, and two persons, j and k. Suppose that Jay is the name of person *j*, and Kay is the name of person *k*. Further suppose that in one situation, person *k* has bowls *a*, *b*, *c*, and *d*, and that person *j* has bowl *e*. Then person *k* gives two of her bowls to person *j*, creating another situation in which person *k* has bowls *a* and *b*, and person *j* has bowls *c*, *d*, and *e*. The intensions of some terms in propositions about this world would be:

3

^Jay:	$[\langle s^{\bullet}\rangle \rightarrow j]$
^Kay:	$[\langle s^{\bullet}\rangle \rightarrow k]$
^bowl:	$[\langle s^{\bullet}\rangle \rightarrow \{a,b,c,d,e\}]$
^has:	$\begin{bmatrix}\langle s_1\rangle \rightarrow \{(j,e),(k,a),(k,b)(k,c)(k,d)\} \\ \langle s_2\rangle \rightarrow \{(j,c),(j,d),(j,e)(k,a)(k,b)\}\end{bmatrix}$

The symbol $\langle s_1\rangle$ is the index for the first situation, and $\langle 2_2\rangle$ is the index for the second situation. $\langle s^{\bullet}\rangle$ is shorthand for all the situations in the model. The first line says that the denotation of the term Jay is the individual object *j*, in all the situations. The third line says that the denotation of the term bowl is a set of objects, $\{a,b,c,d,e,\}$, and this also applies in all the situations. The last entry says that the denotation of has is a set of pairs. In situation $\langle s_1\rangle$, the pairs are (j,e), (k,a), (k,b), (k,c), and (k,d), that is, *j* has *e*, *k* has *a*, *k* has *b*, *k* has *c*, and *k* has *d*. In situation $\langle s_2\rangle$, the pairs are not the same as in $\langle s_1\rangle$. In $\langle s_2\rangle$, *j* has *c*, *j* has *d*, *j* has *e*, *k* has *a*, and *k* has *b*.

To map a propositional representation onto a situation, the variables in the proposition must be assigned values that make the proposition true.

eight bowls; then he lost two of them; now six of them are left." In the second and third sentences, "them" refers anaphorically to the set of eight bowls that Jay had. "Two of them" and "six of them" denote subsets of that set of eight bowls, and "left" indicates that the set denoted by "six of them" is a complement of the set that "two of them" denotes. With linguistic cardinality, a translation includes predicates that denote relations between sets. If there are sets of objects in the task situation, these predicates denote the pairs of sets that have the relations—that is, the set of two lost bowls are a subset of the initial eight bowls, the set of six bowls that are left is another subset of the initial eight bowls, and the two lost bowls and the six remaining bowls are complementary sets.

Another relation between sets that can be cued linguistically is that of disjoint sets, denoted by the term *more*, as in "Kay had four books; then she bought five more books." With linguistic cardinality, "more" can relate the books Kay bought to the books she had by an implicit anaphoric reference.

If competence does not include the principle of linguistic cardinality, then propositional representations will lack references to sets. Individual sentences such as "Jay had eight bowls" will be represented (recall formula *1*), but anaphoric reference such as "of them," "left," and "more" cannot be translated into propositions, because the objects they refer to are not in the representations. If there are physical objects in the situation, or if the child creates a mental model of objects, then sets of those objects provide denotations of the relational terms, and coherent representations can be constructed. This result provides an interesting example of a way in which physical or mental models enable language to be understood when the listener's or reader's lexical knowledge is inadequate in some important ways.

Numerical Comparisons Between Sets. A third meaning of numerals that Greeno et al. (in progress) propose involves reference to differences between sets. Consider a situation with pictures of five birds and two worms. An interviewer says, "Here are some birds and some worms. There are three more birds than worms." In this case "three" is not the cardinality of any specific set of birds. Rather, it is the magnitude of a relation between the set of birds and the set of worms. Understanding the meanings of numerals as properties of relations, rather than only of sets, involves another principle that we call *linguistic set difference*.

Let $\{b_1, \ldots, b_5\}$ denote a set of pictures of five birds, and let $\{a_1, a_2\}$ denote a set of pictures of two worms. Then, with the principles of linguistic cardinality and set difference, a translation of a comparative sentence is:

Short

8 "There are three more birds than worms" $\Rightarrow \exists Y[\forall y(y \in Y \rightarrow \text{worm}(y)) \wedge (\{a_1, a_2\} = Y) \wedge \exists\Delta\exists X[3(\Delta) \wedge (\text{more}(X, Y) = \Delta) \wedge \forall x(x \in X \rightarrow \text{bird}(x)) \wedge (\{b_1, \ldots, b_5\} = X)]]$.

Δ is a variable that takes a relation between sets as its value. Formula 8 says that there is a set Y, the members of Y are worms, Y is the set $\{a_1, a_2\}$. Further, there is a difference relation Δ and a set X; the magnitude of the difference is 3; the difference relation is more (X, Y); the members of X are birds; and X is the set $\{b_1, \ldots, b_5\}$. Assignments of variables that make the sentence true in the situation are $Y = \{a_1, a_2\}$; $X = \{b_1, \ldots, b_5\}$; and $\Delta = \{(\{b_1, \ldots, b_5\}, \{a_1, a_2\})\}$. The intensions of terms are:

9

$^\wedge\underline{\text{bird}}$:	$[<s_1> \rightarrow \{b_1, \ldots, b_5\}]$
$^\wedge\underline{\text{worm}}$:	$[<s_1> \rightarrow \{a_1, a_2\}]$
$^\wedge\underline{\text{more}}$:	$[<s_1> \rightarrow \{(\{b_1, \ldots, b_5\}, \{a_1, a_2\})\}]$
$^\wedge 3$:	$[<s_1> \rightarrow \{\{(\{b_1, \ldots, b_5\}, \{a_1, a_2\})\}\}]$

The denotation of more is the set of pairs of sets in which the first set has more members than the second set. The denotation of *3* is the set of pairs of sets in which the sets differ by three.

If the principle of linguistic set difference is not included in the domain-specific part of competence, a translation of "three more birds" might be obtained using the meanings of *three* and *more* that we discussed previously. These meanings are inappropriate for comparison of sets; they would apply correctly for a text such as "There are two birds on the bird feeder; then some more birds came and there were three more birds on the bird feeder." If "There are three more birds than worms" is translated using only the principle of linguistic cardinality, the result is:

10 "There are three more birds than worms" $\Rightarrow \exists Y[\forall y(y \in Y \rightarrow \underline{\text{worm}}(y)) \wedge (\{a_1, a_2\} = Y) \wedge \exists X[3(X) \wedge \underline{\text{more}}(X, Y) \wedge \forall x(x \in X \rightarrow \underline{\text{bird}}(x)) \wedge (\{b_1, \ldots, b_5\} = X)]]$.

This translation is analogous to the translation of "Kay bought three more books," where "more" means that the books Kay bought are different from those she had before. Of course, *10* is semantically incorrect; it says that X has cardinality three, but it identifies X as a set with five members.

Setting Goals for Inferences

In experimental tasks, children often show their understanding by answering questions for which they have to make inferences. In this section we turn to the process of setting goals for actions that result in inferences. We be to add a reference to the set of objects to the representation so that 5 could be included as a property of something.

Young children may not interpret "How many?" in cardinal terms. They may think it means "Count," or "Say some number word." In the latter case, the best candidate is the last said, simply because it is most likely to be the one in short-term memory. This and the second option aforementioned emphasize the possibility that the children had not yet learned a principle of linguistic cardinality; the first option emphasizes the role of conversational rules in interpretative competence. Research is needed to choose between these. Still, these are all accounts that implicate limits on how children use language, be it number specific or not. Once again then, our analysis shows how children could have implicit knowledge of the counting principles before they developed more extensive knowledge of them.

In a previous section we discussed one of Markman's effects of varying the use of class terms and collection terms. We turn now to our account of why this variation in language points to an explanation of recounting with reference to interpretative competence. In one experiment, Markman (1979) presented sets of toys, such as blocks, to 3- and 4-year-old children in two conditions. To children in the *class condition* the experimenter said, "Here are some blocks; count the blocks," and to children in the *collection condition* the experimenter said, "Here is a pile of blocks; count the blocks in the pile." After the child counted the blocks, the experimenter asked "How many are there?" In the class condition in Markman's study, children recounted on about one-half of the trials, whereas when collection terms were used, recounting occurred on fewer than 10% of the trials. In a subsequent study by Fuson et al. (1985), a different group of 3- and 4-year-olds answered "How many?" by giving the last numeral used in counting on about three-fourths of the trials when class terms were used, and on 60% of the trials when collection terms were used. We think the discrepancy in findings is due to the ambiguity of the situation. To show why, we first discuss Greeno et al.'s analysis of the meaning of collection terms.

In analyzing the meaning of ***a pile of blocks***, Greeno et al. (in progress) use the concept, developed by Massey (1976), of composite objects, which are not the same as sets. Composite objects contain other objects as parts, whereas sets contain individual objects as members. Examples that clarify this distinction are sentences such as "Tom, Dick, and Harry carried the piano upstairs." A group of men did the carrying, acting as a whole, and the logic of sets fails to capture that kind of relation.

Massey used the notation +() to denote composite objects, so that +(Tom, Dick, Harry) denotes the composite object, made up of Tom, Dick, and Harry, not the set whose members are Tom, Dick, and Harry, which would be denoted {Tom, Dick, Harry}. Let the blocks shown to a

child be denoted $a,b,c,d,$ and e. Also, assume that competence does not include the principle of linguistic cardinality. If it did, then a set would be represented and the collection term would probably be redundant. Then, using Massey's notation, the translation of "Here is a pile of blocks" is:

12 "Here is a pile of blocks" $\Rightarrow \exists u[\underline{\text{pile}}\ (u) \wedge$
$\exists v \exists w \exists x \exists y \exists z[\underline{\text{block}}\ (v) \wedge \ldots \wedge \underline{\text{block}}\ (z) \wedge$
$\underline{\text{made-of}}\ (u\ v) \wedge \ldots \wedge \underline{\text{made-of}}\ (u,z) \wedge$
$(+(a,b,c,d,e) = u)]$;

that is, there is an object u, such that u is a pile, and there are objects v, w, x, y, and z, and objects $v, \ldots, z$ are blocks, and u is made of $v, \ldots, z$, and u is the composite object $+(a,b,c,d,e)$. Assignments that make the formula true are $v = a$, $w = b$, $\ldots$, $z = e$, and $u = +(a,b,c,d,e)$. Intensions of terms in the representation are:

13 $^\wedge\underline{\text{block}}$: $[<S_1> \rightarrow \{a,b,c,d,e\}]$
$^\wedge\underline{\text{pile}}$: $[<S_1> \rightarrow +(a,b,c,d,e)]$
$^\wedge\underline{\text{made-up}}$: $[<S_1> \rightarrow \{(+(a,b,c,d,e), a), (+(a,b,c,d,e), b), \ldots, (+(a,b,c,d,e), e)\}]$

Formula *12*, with a collection term, does not include reference to the set of blocks, and therefore modifying *12* to include the result of counting would not be as easy as modifying *5* to get *11*. On the other hand, the representation does include the composite objects, and the parts of that object are the members of the set that is counted. This might encourage *some* children to create a representation of the set and add it to the propositional representation. The matter is subtle, however, and untangling it probably will require research on the general question of children's understanding of terms that denote composite objects in contexts other than those involving numbers. Understanding the conditions in which children tend to recount sets also requires clarifying the effect of the social and conversational context involved when an experimenter asks a question directly about the result of their counting.

Questions About Amounts of Difference. Another finding that implicates the role of language in reasoning about quantity is Hudson's (1983) result, showing that, if young children are asked "How many more . . . are there than . . . ?", they often respond with the number in the larger set, but, if they are asked "How many . . . won't get a . . . ?", they usually give the amount of the difference between the sets.

The principle of linguistic set difference provides an interpretation of Hudson's finding. First, consider the representation of the question when competence includes linguistic cardinality and set difference. If the pictures with linguistic cardinality included in competence, the sentences corresponding to "These are Jay's soldiers" and "These are Kay's soldiers" are translated to:

18 "There are some soldiers such that Jay has them" ⇒
$\exists X[\forall x(x \in X \rightarrow \underline{\text{soldier}}\ (x)) \wedge \forall x(x \in X \rightarrow \underline{\text{has}}\ (\underline{\text{Jay}}, x)) \wedge (\{a_1, \ldots, a_7\} = X)]$.

"There are some soldiers such that Kay has them" ⇒
$\exists Y[\forall y(y \in Y \rightarrow \underline{\text{soldier}}\ (y)) \wedge \forall y(y \in Y \rightarrow \underline{\text{has}}\ (\text{Kay}, y)) \wedge (\{b_1, \ldots, b_7\} = Y)]$.

Assignments of variables are $X = \{a_1, \ldots, a_7\}$ and $Y = \{b_1, \ldots, b_7\}$. Suppose that Jay's soldiers are placed on [one] card C_1 and Kay's soldiers are placed on [another] card C_2. Then, the intensions are:

19

$^\wedge\underline{\text{soldier}}$:	$[<s_1> \rightarrow \{a_1, \ldots, a_7, b_1, \ldots, b_7\}]$
$^\wedge\underline{\text{Jay}}$:	$[<s_1> \rightarrow C_1]$
$^\wedge\underline{\text{Kay}}$:	$[<s_1> \rightarrow C_2]$
$^\wedge\underline{\text{has}}$:	$[<s_1> \rightarrow \{(C_1, a_1), \ldots, (C_1, a_7), (C_2, b_1), \ldots, (C_2, b_7)\}]$

Then a question is asked: 'Which is more: Jay's soldiers or Kay's soldiers, or are they the same?" Again avoiding analysis of question syntax, Greeno et al. analyze the sentences: "Jay has more soldiers than Kay," "Kay has more soldiers than Jay," and "Jay has the same soldiers as Kay," with "the same" translated as equal and interpreted as having an equal number. If the principle of linguistic-set difference is included along with linguistic cardinality, the translation of comparative sentences is:

20 "Jay has more soldiers than Kay" ⇒
$\exists Y[\forall y(y \in Y \rightarrow \underline{\text{soldier}}\ (y)) \wedge \forall y(y \in Y \rightarrow \underline{\text{has}}\ (\underline{\text{Kay}}, y)) \wedge$
$\exists \Delta \exists X[(\Delta = \underline{\text{more}}\ (X, Y)) \wedge \forall x(x \in X \rightarrow \underline{\text{soldier}}\ (x)) \wedge$
$\forall x(x \in X \rightarrow \underline{\text{has}}\ (\text{Jay}, x))]]$,

and the corresponding formulas with more (Y, X) and equal (X, Y). When the interpreter sets goals based on these sentences, the meaning of Δ as a numerical difference between sets results in setting goals to compare the cardinalities of the sets, and this leads to the correct answer.

If competence does not include linguistic set differences but includes cardinality, the demonstrative sentences lead to the same representation as *18* and *19*, but the comparative sentences are:

21 "Jay has more soldiers than Kay" ⇒
$\exists Y[\forall y(y \in Y \rightarrow \underline{\text{soldier}}\ (y)) \wedge \forall y(y \in Y \rightarrow \text{has}\ (\text{Kay}, y)) \wedge$
$\exists X[\text{more}\ (X, Y) \wedge \forall x(x \in X \rightarrow \underline{\text{soldier}}\ (x)) \wedge$
$\forall x(x \in X \rightarrow \underline{\text{has}}\ (\text{Jay}, x))]]$,

and the corresponding formulas with more (Y, X) and equal (X, Y). Recall that the meaning of more in this reading is the same as "other than," which would make either "Jay has more soldiers than Kay" or "Kay has more soldiers than Jay" correct, and would make "Jay has the same soldiers as Kay" incorrect, assuming that "the same" is understood as meaning identical soldiers. On the other hand, the situation would be somewhat confusing for a child because a presupposition of the question is that only one of the three alternatives is correct. We can only guess how a child might respond, but one possibility could be to consider the numbers of objects in the sets and answer on the basis of those.

If competence does not include linguistic cardinality, then the representations of "These are Jay's soldiers" and "These are Kay's soldiers" include references to individual objects, rather than to sets. Lacking reference to sets of soldiers could correspond to the stage of understanding that Piaget called graphical collections, which are spatially defined and might well lead to a spatial interpretation of "more" in the question "Which is more?" Of course, the sets of objects are present, and, if the child interprets more and equal numerically, he or she could set goals for counting the sets of objects, and the correct answer could be given without having references to sets in the propositional representation.

Markman (1979) found that use of collection terms improved the performance of preschool children on number conservation tasks. If we assume that collection terms such as *army* denote composite objects in Massey's (1976) sense, and let the two armies be denoted $+(a_1, \ldots, a_7)$ and $+(b_1, \ldots, b_7)$, then the translations of sentences corresponding to "This is Jay's army" and "This is Kay's army" are:

22 $\exists u[\underline{\text{army}}\ (u) \wedge \text{has}\ (\underline{\text{Jay}}, u) \wedge (+(a_1, \ldots, a_7) = u)]$;
$\exists v[\underline{\text{army}}\ (v) \wedge \text{has}\ (\underline{\text{Kay}}, v) \wedge (+(b_1, \ldots, b_7) = v)]$.

The comparative sentence corresponding to "Jay's army is more than Kay's army" is:

23 $\exists v[\underline{\text{army}}\ (v) \wedge \text{has}\ (\underline{\text{Kay}}, v) \wedge \exists u[\underline{\text{more}}\ (u, v) \wedge$
$\underline{\text{army}}\ (u) \wedge \underline{\text{has}}\ (\underline{\text{Jay}}, u)]]$.

It would not be surprising if references to the composite objects would lead some children to consider the comparisons numerically. However, it would also not be surprising if they did not. A case could be made that understanding armies, piles of blocks, and so on as composite objects might even emphasize their spatial extents, rather than their cardinalities, but that would predict the opposite effect from the one Markman (1979) obtained. As with the phenomenon of recounting, understanding of the role of collective nouns on number conservation ~~of~~ probably requires reunderstanding of a domain unless their performance is consistently correct and unless they succeed on a wide variety of tasks that are reasonably defined as relevant to that domain. Note that the conclusion also presumes that the conceptual competence hypothesis predicts essentially no variability in performance, either within or across tasks in a domain.

Although it is true that variability in performance is readily explained by the no-principle model, it is *not* true that the principle-first model is ruled out by this criterion. This criterion by itself is actually neutral in its ability to discriminate between the two classes of models. For, to assume that the presence of conceptual competence in a domain guarantees successful performance on any tasks therein is to maintain that conceptual competence is sufficient for the successful execution of the kind of behavior required in any given task setting. But this is not the case. First, conceptual competence does not contain recipes for successful behaviors. The fact that infants can respond categorically to speech sounds before they can make these sounds helps make this point. What conceptual competence does do is provide those constraints the planner must honor if it is to generate a successful plan of action. In the case of counting, additional cognitive components, including domain-linked knowledge, procedural competence, and interpretative competence, are all needed to derive competent plans for action. Once derived, these plans have to be executed without error and, as we noted earlier, this too is a matter of no small consequence. Hence, competence models like ours are consistent with—indeed predict—variability in just the way it is observed. We develop this conclusion and then turn to the question of how then to distinguish between the two models.

To review, conceptual competence does not contain instructions for putting the counting principles into practice. It provides domain-relevant constraints that the planner must honor for it to assemble a potentially successful plan for counting. But even should the planner take all the constraints of conceptual competence in the domain into account, there is still no guarantee that a complete plan can be assembled for a given task in a given setting. For this to happen, the planner must also take into account certain classes of variables that contribute to the correct interpretation of the setting and task—including some that are domain specific and some domain general.

Domain-specific variables are ones like the quantitative vocabulary used in a problem or the instructions, the props that can be used to solve the problem, and the type or arrangement of items. Domain-general variables include the kind of social setting for the task as well as interpretations of the kinds of conversations that take place in that setting. If all the task requirements are correctly interpreted, be they domain specific or domain general, and, if all these are honored in concert with the constraints of the counting principles, then the stage is set for the assembly of a successful plan. But success at this point does not guarantee accurate performance.

On the assumption that a successful plan is actually generated, the stage is finally set for the potential execution of a performance. But even now there is still room for error. Just as speech errors can occur as a consequence of output problems that occur in the postplanning stage (Fromkin, 1973) so errors are possible in the execution stage of a successful plan. As indicated earlier, this is especially so in the initial efforts at execution.

The point should be clear now: Our model of how individuals generate competent plans of action is at least as consistent with the fact that the young produce errors and variable levels of performance as are association theory accounts. Additionally, our model offers a classification scheme of the sources of variability. One can ask whether failure or errors occur because a child lacks the requisite conceptual competence, has made an interpretative error, has a faulty planner, or encounters problems in the execution of an acceptable plan. Before concluding that an error on the output side means there is no conceptual competence, one must consider whether any of these other contributors to the planning and execution processes are the culprits. In addition to conceptual competence, there are other conditions for correct performance. Hence, the facts of variability in their own right do not rule out a principle-first account of counting—or, for that matter, any other domain.

Ways to Distinguish the Two Classes of Models

If variability per se cannot distinguish the principle first from the principle after, what criteria can? We have introduced some in various places throughout the chapter. Here, we pull these together and focus on the kinds of analyses that might be appropriate.

As indicated, a principle-first model implies generativity. If children are able to generate some plans of action they have not used before, we can grant them implicit knowledge of conceptual competence. In the abstract, this clear prediction separates the two models. For there would be no account of how children could succeed if they did have some principled understanding from which to generate novel solutions. In actuality, novel tasks involve novel settings and children could fail because the setting is problematic. Still, there seems a clear prediction to make. And if either lack of familiarity with, or knowledge of, a setting presents an impediment, then task variations designed to let children come to understand the critical ingredients of the setting also should reveal their competence. Like Wilkinson (1984), Gelman and Meck (1986) found that one way to do this is to let children answer a question a second time. To illustrate, consider one of the Gelman and Meck (1986) tasks.

evaluating the different predictions of different models if we are to avoid accepting or rejecting models on the grounds of criteria that do not allow us to discriminate between them.

The Competence–Performance Distinction Reconsidered

Some will think we have complicated the notion of competence. For there is the view that competence models are nothing but implicit knowledge structures for a given domain, be the domain syntax, space, logic, or ornithology. We do not take this perspective but prefer instead the idea that competence in a domain is being able to generate competent plans of action that honor the constraints of the principles of knowledge in that domain. Otherwise the plans could not yield examples of behaviors that are in the equivalence class of the target area. For procedural competence to generate all and only those plans that meet this criterion requires that consider more than the constraints of conceptual competence. It also must be constrained by domain-relevant interpretations of the setting. This could not happen if conceptual competence were not driving the attention to and learning about those aspects of the environment that are suitable stimuli.

What emerges from the present analysis of conceptual competence is the idea that principles provide guidelines for behavior. They constrain the range of stimuli one should attend to, the kinds of things one should learn to expand knowledge of a domain, and the class of behaviors one can suitably organize to meet goals of action that are recognized as relevant to the domain in question. They do not, however, come packaged with recipes for successful outputs.

With the foregoing in mind, we propose that traditional views of the competence–performance distinction be modified to deal with the fact that successful behaviors depend on competent plans of action. We believe theories of the mind should be theories of how the mind governs action as well as pure thoughts; they have to deal with the fact that actions are often part and parcel of thinking. This is especially true in the problem-solving domain, and, hence, models of competence that fail to address the question of how conceptual competence links to behavior must be viewed as incomplete. Thus, it hardly makes sense to pit competence against performance. The focus should be the difference between domain-specific and domain-independent variables as contributors to competent plans for performance. Domain-specific competence is a *sine qua non* for successful problem solving with a domain. Although necessary, it is not sufficient. We have seen that other competences, ones we refer to as domain general, are also involved in the production of a competent plan. Given this, the question is what contributes to the successful execution of that plan and thus successful performance.

short

TOWARD A MODEL OF ACTIVE LEARNING: SOME SPECULATIONS

Recent discussions have highlighted the way the young participate actively in their own learning (Brown, Bransford, Ferrara, & Campione, 1983; Gelman & Brown, 1986; Resnick, 1986). Young learners often ignore what an adult tries to get them to do; they focus on what they find interesting; they monitor, on their own, their progress through a solution; they self-correct, or at least start a trial over again without obvious external feedback; and they keep at a problem until they find a better solution than one they have achieved, even if this means going from an error-free solution to one that is error prone. Especially compelling examples of such self-initiated and active involvement are provided by Karmiloff-Smith and Inhelder's (1974/75) case study of children learning to balance blocks and Brown and Reeve's (1986) analyses of how toddlers become progressively more sophisticated in stacking cups that are ordered in size. Children persist at these activities and move through more and more sophisticated levels of solutions without any external feedback. There is no denying the obvious supporting role that others can play. But any model of learning must also deal with the fact that children do not simply absorb the material we put in front of them. A model must account for their selective attention and motivation, their self-generated learning activities, and their ability to make progress without benefit of others telling them the solutions. Analyses of competence like ours contribute some essential components of such an account.

Principles Direct Attention and Support Organized, Coherent Storage of Data

Earlier we proposed that infants attend to number-relevant information from displays because their attention is directed by implicit knowledge of at least some components of pertinent mathematical principles. Gelman (1986) developed this argument, that is, that domain-specific principles can help the young organize the search of their environment for domain-relevant inputs. The idea is that principles provide clues regarding the way environments are used to serve a particular goal or function. Because numerals are used by others in a way that satisfies the counting principles, and because color terms and the alphabet entries are not, the child who has

PP1 (30).max

than one is present at a given time; the same count word cannot be used under such a condition.

Our claim is not that children are unable to learn to sort these different classes of verbal materials if they lack principles for identifying labels and count words. Without them, however, learning should depend fully on an associative process and, therefore, be more difficult than principle-aided efforts. Further, the initial ability to use a given class of words should be more variable across children. Landau and Gleitman (1985) and Gelman and Meck (1968) offer preliminary data to support such an account of why young children have difficulty learning to use color words, words that are surely ubiquitous in their environments.

Our account of how children learn the count words is incomplete. For children have to do more than identify correctly the class of count words and use the first three in the list. They have to master the list. Because the learning of long lists is a formidable task even for adults, we might expect that mastery is attained over a long course. This is likely to be especially true in English because the base rule that serves to generate entries is not transparent until the list gets up to the 20s and 30s (Fuson & Hall, 1983). Given the mind's penchant for imposing its own order on difficult inputs like lists, we might even expect young children to fix on somewhat idiosyncratic lists when they count. Errors, like saying the number after 29 is 20/10 or after 999 is 990/10, occur often enough to get reported (Hartnett & Gelman, 1987; Miller & Stigler, 1987); so do the private or idiosyncratic lists like 1-2-11-8 that some 2-year-olds use (the one here was brought to our attention by a friend on behalf of her young grandaughter).

Despite the fact that the idiosyncratic lists are rule governed and/or serve the requirements of the stable-ordering principle, their continued use will produce communication problems. Rule governed or not, they have to be discarded and replaced with the conventional count lists, let alone the kinds of linguistic representations we discussed earlier. Presumably, the desire of both adults and children to communicate with each other serves as one contributor to this development. For, on the basis of volunteered reports to Gelman, adults are bewildered by the child who looks at a set of four objects and insists there are eight or refuses to include "two" in his count list. (Indeed, we often discover another idiosyncratic count list when the claim that the child cannot count yet is discussed.) On the assumption that children give up what does not communicate and try to find what does, they eventually should adopt the conventional list, especially because it is likely that the adult who thinks the child has made an error will provide it, just as they correct children who mislabel (R. Brown, 1973).

Our suggestions regarding the acquisition of a domain-linked partitioning schema also apply to how the count words start to take on meaning, even if they do not become the subject of a conversation. When actions

like pointing and number word use meet the constraints of counting competence, they can be reinforced because they are consistent with the requisites the planner has to monitor. If they meet these, then there is no need for the child to count in a different way. As a result, the very same solution and words might be used again, and again. Such an account allows that what look like conventional counting procedures might even be invented and added to the child's knowledge about counting in a way that overlaps with that of the more mature counter. In other words, whether or not someone reinforces a child's use of conventional procedures, the simple fact that he or she honors constraints can serve to reinforce their use. We note that a similar analysis could be developed for the acquisition of mathematical terms and the meaning of numerals. Indeed, it is implicit in the different versions of the models for understanding previously developed. This argument is made explicit in the next section.

Constraints Provide the Planner With Monitoring Potential

Young children have a ubiquitous tendency to persist until they get something right. To do this they must have capacity to monitor (of course without awareness) the relationship between their outputs and the target they aim for. Our model of planning provides a hint as to how to characterize this. Because the planner has to determine whether a chosen procedure meets the requisites dictated by the principles, it can serve as a potential source of feedback to children learning to count or solve a novel problem. For, if the requisite conditions are not met, the planning effort or execution of a plan can be rejected or aborted. This means that the child can start again—without being told to do so by an external agent.

The plans that we assume are constructed from competence schemata could serve as cognitive templates for actions. The importance of a cognitive schema in an early stage of skill acquisition was recognized by Bruner and Koslowski (1972) and Fitts (1964) and has been confirmed in recent research by Pirolli (1987). A learning mechanism that is responsive to the extent to which a plan and the output from it honor requisites of competence could, in turn, use the template (more generally, the representation) to monitor the success or failure of performance. Such representations could serve three functions.

First, the plan for an action sequence might be incomplete, leaving some actions in the sequence unspecified, but specifying the goal or subgoal that the missing actions should achieve. This would enable trials of actions that could be evaluated according to their local effects and obviate the need to wait for the outcome of the complete action sequence and an analysis to determine whether a given component was successful or not. Second, plans based on competence schemata could serve as a basis for confirmation that be counted has to be partitioned into the subset that has been counted and the subset that remains to be counted, and counting is complete only when all members of the set are in the counted subset. The important fact is that, in counting, the objects have to be treated as a set, but that feature is incorporated in the constraints on actions and is therefore embedded in the structure of the actions rather than explicitly available as a feature that is accessible to other reasoning. The suggestion that emerges is that acquisition of word meanings with references to sets could involve abstraction and explication of features that are implicit in the actions and plans of counting. In an early stage, numeral terms might denote counting procedures that include set-theoretic features implicitly, and later, the denotations of numeral terms would include those features explicitly.

Another transition would involve acquiring the meaning of numerals as denoting differences between sets in contexts involving comparisons. A hypothesis about learning the principle of linguistic set differences is suggested by Greeno et al.'s (in progress) hypothesis that understanding set differences involves a new meaning of numerals. Understanding about differences between sets could develop after children have learned about differences between cardinal numbers as part of their instruction in arithmetic, where they learn to add and subtract. This conjecture is consistent with the fact that children come to understand such sentences as "There are three more birds than worms" after they have studied arithmetic in school for some time. In data collected by Riley and reported by Riley, Greeno, and Heller (1983), word problems involving questions of "How many more?" and "How many less?" were solved correctly by 80% of the children who were near the end of second grade, but by only 25% of the children who were near the end of first grade.

CONCLUSIONS

The studies and analyses reviewed in this chapter contribute to a characterization of young children's understanding of concepts and principles of mathematics and provide some suggestions for a theory of learning in mathematics and other subject-matter domains. We have not developed suggestions here that directly apply to the practice of instruction. The results and ideas that we have developed, however, raise some issues about instruction that we mention briefly in conclusion.

Assessment of Understanding. The accumulating results of research on children's understanding in mathematics and other domains has the clear implication that, if tests of children's knowledge are limited to their performance of procedures that they have been taught, we will fail to obtain data that adequately reflect their understanding. It is now a firm

conclusion for many domains that children understand significant principles that are not reflected in their performance of the tasks that are commonly used in instruction and standard tests. It is important for instructional research to develop and analyze methods of assessment that more adequately capture children's understanding. A significant feature of performance that reflects understanding is its *generative* character; to assess understanding, we need to present opportunities to respond to novel situations. Research and development on new methods of assessment will require major advances in basic scientific knowledge and theory about the characteristics of that understanding, beyond results that are already available. Still, theoretical concepts and methods are available to support the research that is needed.

Our analyses of conceptual understanding build representations upon representations upon representations of beginning levels of conceptual competence. They highlight the need to avoid assessments of an all-or-none kind of understanding. Instead, they point to assessments that identify the level of understanding a child already has as well as those he or she may be reaching (cf. A. Brown, chap. 13 in this volume). This is no less true regarding the understanding of number words than it is regarding the understanding of advanced principles of mathematics. The reason for this emerged in our analyses of linguistic principles of domain-specific competence. An amazing amount of development goes into the achievement of a linguistic version of a mathematical principle.

Instruction that Activates Understanding. It is a major problem that many students learn procedures for manipulating symbols without relating them to the underlying concepts and principles. (The now well-known "bugs" of elementary arithmetic, see Brown & Burton, 1978, provide striking examples.) Efforts to teach children the meanings of the procedures, for example, by using manipulative materials, such as place-value blocks, have not been as successful as might be expected (e.g., Resnick & Omanson, in press). The results of research on competence suggests a dimension of meaningful learning that has not been emphasized. It should be possible, at least in principle, to analyze the problem of learning as a transformation of competence. We suggest that children's competence can play a crucial role in supporting learning in subject-matter domains by guiding attention to relevant features of the environment and recruiting new procedures that are linked meaningfully to general structures of competence. Apparently this does not happen in much school instruction, particularly in mathematics. Designs of curriculum and instruction that connect new information and tasks with the concepts and principles of the domain that children already understand, and that extend and explicate that understanding, would constitute major improvements. The kinds of

In N. Smelser & D. Gerstein (Eds.), *Discoveries and trends in behavioral and social sciences* (pp. 175-207). Commission on Behavioral and Social Sciences and Education. Washington, DC: NRC Press.
Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number*. Cambridge, MA: Harvard University Press.
Gelman, R., & Meck, E. (1983). Preschoolers' counting: Principles before skill. *Cognition, 13*, 343-359.
Gelman, R., & Meck, E. (1986). The notion of principle: The case of counting. In J. Hiebert (Ed.), *The relationship between procedural and conceptual competence* (pp. 29-57). Hillsdale, NJ: Lawrence Erlbaum Associates.
Gelman, R., Meck, E., & Merkin, S. (1986). Conceptual competence in the numerical domain. *Cognitive Development, 1*, 1-29.
Ginsburg, H., & Russell, R. L. (1981). Social class and racial influences on early mathematical thinking. *Monographs of the Society for Research in Child Development, 46* (6, Serial No. 193).
Glaser, R. (1976). Cognitive psychology and instructional design. In D. Klahr (Ed.), *Cognition and instruction* (pp. 303-316). Hillsdale, NJ: Lawrence Erlbaum Associates.
Goffman, E. (1974). *Frame analysis*. Cambridge, MA: Harvard University Press.
Greeno, J. G. (1984). Forms of understanding of mathematical problem solving. In S. G. Paris, G. M. Olson, & H. W. Stevenson (Eds.), *Learning and motivation in the classroom*. Hillsdale, NJ: Lawrence Erlbaum Associates.
Greeno, J. G., & Johnson, W. (1984). *Competence for solving and understanding problems*. Paper presented at the International Congress of Psychology, Acapulco, Mexico.
Greeno, J. G., Johnson, W., & Goldberg, A. (in progress). *Competence for understanding language and reasoning with mental models*.
Greeno, J. G., Riley, S., & Gelman, R. (1984). Conceptual competence and children's counting. *Cognitive Psychology, 16*, 94-143.
Grice, H. P. (1975). Logic and conversation. In P. Cole & J. L. Morgan (Eds.), *Speech acts: Syntax and semantics* (Vol. 3). New York: Academic Press.
Hartnett, P., & Gelman, R. (1987, April). *Inducing further numerical principles given some knowledge of principles of counting and addition*. Paper presented at the meeting of the Society for Research in Child Development, Baltimore.
Heath, S. E. (1981). Questioning at home and school: A comparative study. In G. Spindler (Ed.), *Doing ethnography: Educational anthropology in action*. New York: Holt, Rinehart, & Winston.
Hudson, T. (1983). Correspondences and numerical differences between disjoint sets. *Child Development, 54*, 84-90.
Hull, C. L. (1937). Mind, mechanism and adaptive behavior. *Psychological Review, 44*, 1-32.
Hymes, D. (1967). Models of the interaction of language and social setting. *Journal of Social Issues, 23*, 2 8-28.
Inhelder, B., & Piaget, J. (1959/1964). *The early growth of logic in the child*. New York: Harper & Row.
Inhelder, B., Sinclair, H., & Bovet, M. (1974). *Learning and the development of cognition*. Cambridge, MA: Harvard University Press.
Johnson-Laird, P. N. (1983). *Mental models: Towards a cognitive science of language, inference and consciousness*. Cambridge, MA: Harvard University Press.
Karmiloff-Smith, A., & Inhelder, B. (1974/75). If you want to get ahead, get a theory. *Cognition, 3*, 195-212.
Kellman, P. J., & Spelke, E. S. (1981). Perception of partly occluded objects in infancy. *Cognitive Psychology, 15*, 483-524.

Kintsch, W., & Greeno, J. G. (1985). Understanding and solving word arithmetic problems. *Psychological Review, 92*, 109-129.
Lampert, M. (1986). Knowing, doing and teaching multiplication. *Cognition and Instruction, 3*, 305-342.
Landau, B., & Gleitman, L. (1985). *Language and experience: Evidence from the blind child*. Cambridge, MA: Harvard University Press.
Mandler, J. M. (1983). Representation. In J. H. Flavell & E. M. Markman (Eds.), *Handbook of child psychology: Vol. 3. Cognitive development* (4th ed., pp. 420-494). New York: Wiley.
Mandler, J. M. (1984a). *Stories, scripts and scenes: Aspects of schemata theory*. Hillsdale, NJ: Lawrence Erlbaum Associates.
Mandler, J. M. (1984b). Representation and recall in infancy. In M. Moscovitch (Ed.), *Infant memory*. New York: Plenum.
Markman, E. M. (1979). Classes and collections: Conceptual organization and numerical abilities. *Cognitive Psychology, 11*, 395-411.
Markman, E. M. (1986). How children constrain the possible meanings of words. In U. Neisser (Ed.), *The ecological and intellectual basis of categorization*. Cambridge, England: Cambridge University Press.
Markman, E. M., & Hutchinson, J. E. (1984). Children's sensitivity to constraints on word meaning: Taxonomic vs. thematic relations. *Cognitive Psychology, 16*, 1-27.
Markman, E. M., & Seibert, J. (1976). Classes and collections: Internal organization and resulting holistic properties. *Cognitive Psychology, 8*, 561-577.
Massey, G. J. (1976). Tom, Dick, and Harry, and all the king's men. *American Philosophical Quarterly, 13*, 89-107.
Miller, K., & Stigler, J. (1987). *Counting in Chinese: Cultural variation in a basic cognitive skill*. Unpublished manuscript, University of Texas at Austin.
Piaget, J. (1951). *Play, dreams and imitation in childhood*. New York: Norton.
Piaget, J. (1970). Piaget's theory. In P. H. Mussen (Ed.), *Carmichael's manual of child psychology* (Vol. 1). New York: Wiley.
Piaget, J. (1975). *L'équilibration des structures cognitives: Problème central du développment*. Études d' épistemologie génétique (Vol. 33). Paris: Presses Universitaires de France.
Pirolli, P. L. (1987). A model of purpose-driven analogy and skill acquisition in programming. *Proceedings of the Cognitive Science Society Conference*.
Pirolli, P. L., & Anderson, J. R. (1985). The role of learning from examples in the acquisition of recursive programming skills. *Canadian Journal of Psychology, 39*, 240-272.
Premack, D. (1976). *Intelligence in ape and man*. Hillsdale, NJ: Lawrence Erlbaum Associates.
Premack, D., Woodruff, G., & Kennel, K. (1978). Conservation of liquid and solid quantity in chimpanzee. *Science, 202*, 991-994.
Resnick, L. B. (1976). Task analysis in instructional design: Some cases from mathematics. In D. Klahr (Ed.), *Cognition and instruction* (pp. 51-80). Hillsdale, NJ: Lawrence Erlbaum Associates.
Resnick, L. B. (1986). Cognition and instruction: Recent theories of human competence and how it is acquired. In B. L. Hammonds (Ed.), *Psychology and learning: The master lecture series* (Vol. 4 pp. 123-187). Washington, DC: American Psychological Association.
Resnick, L. B., & Omanson, S. F. (in press). Learning to understand arithmetic. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 3). Hillsdale, NJ: Erlbaum Associates.
Riley, M. S., Greeno, J. G., & Heller, J. I. (1983). Development of children's problem-solving ability in arithmetic. In H. P. Ginsburg (Ed.), *The development of mathematical thinking*. New York: Academic Press.