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EARLY UNDERSTANDINGS OF NUMBERS: PATHS OR BARRIERS TO THE CONSTRUCTION OF NEW UNDERSTANDINGS?

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Abstract

We propose that concept development is facilitated when existing conceptual structures overlaps with the structure of the to-be-learned data. When the new inputs do not map readily or are inconsistent with available mental structures, the risk is high that the data will be misinterpreted as examples of what is known. Results of two kinds of studies with children aged 5 to 7 years provide support for these predictions. Children's knowledge of counting and addition facilitated their acquisition of a concept that is not taught in school, the Successor Principle, that is that every natural number has a successor. In contrast, even the eldest children could not rank order correctly numbers that contained fractional notations. The several kinds of solutions invented by the children were based either on classification strategies or a natural number ordering rule.

Introduction

Our account of cognitive development (e.g. Gelman, 1993; Gelman & Williams, 1998) shares several key assumptions with Piaget's and other constructivist theories of mind. These are that: (1) knowledge acquisition involves the construction of structural represen-

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tations; (2) novices actively participate in their own cognitive development; and (3) existing mental structures are used to find, interpret, create and store data whose organization overlaps with available knowledge structures. Together these assumptions have implications for a constructivist theory of the learning processes involved in cognitive development. Children's ability to self-initiate the use of their knowledge structures is fundamental here. It can lead them to self-generate learning trials. This means they also can attend to relevant data, that is, data whose organization is consistent with existing mental structures, and ignore irrelevant data, that is, items in the environment that are not consistent with what they know. Finally, their structures can store assimilated data in a coherent way, the effect of which is the build-up of domain-relevant knowledge.

Although such learner-initiated cognitive activities can support knowledge acquisition, their existence complicates our effort to achieve an account of the mechanisms involved in learning. As long as inputs are consistent with what is known, then novice's active participation in their learning can facilitate knowledge acquisition. But when available mental structures are not consistent with the inputs meant to foster new learning, such self-initiated interpretative tendencies can get in the way. In the absence of a structural map between what is known and what is to be learned, novices might ignore the input. They also can select or even create inputs that are erroneous from the expert's viewpoint but not theirs. Given that the active application of our mental structures can lead us to interpretative "errors", what is known need not always serve as a bridge to later learning. Indeed, what is known can be a barrier to later learnings.

Children's early knowledge about numbers can bridge their further learning about numbers if the structure(s) of what they know are consistent with those of the to-be-learned understandings. For example, knowledge of the counting principles can facilitate learning that the written numerals are marks on paper that represent cardinality and that different numerals are ordered. However, given structural inconsistency between what is known and what is to-be-known, novel inputs might be treated as examples of what is already known and therefore misinterpreted (see also Bassok & Holyoak, 1993). Below we illustrate these differential effects in two sets of studies with kindergarten and elementary school children in the early grades.* One set involved our efforts to teach the successor principle, the idea that every natural number has a successor; the other focused on how children interpret inputs about fractions. It is noteworthy that school children of the age range in this study are taught about fractions but not the successor principle. This difference regarding the early math school curriculum has much significance for us.

Children start kindergarten with the idea that numbers are what "one gets when one counts collections of things". They have moved towards using count words and other symbols in ways that are numerically meaningful and can succeed on some arithmetic and problem-solving tasks with verbal instructions and verbal or written symbolic instantiations of cardinal numerosities and the operations of addition and subtraction (Fuson, 1992; Gallistel & Gelman, 1992; Gelman & Gallistel, 1978; Gelman & Meck, 1992; Gelman, Meck & Merkin, 1986; Groen & Resnick, 1977; Nunes & Bryant, 1996; Siegler & Robinson, 1982; Sophian, 1995). Gallistel and Gelman (1992) propose that learning is facilitated when there can be a structural mapping between what is known about counting and arithmetic

*Kindergartens are part of the regular United States elementary school system, although attendance is not universally obligatory. Children are 5 years of age if they attend these.

and to-be-learned mathematical representations. This condition holds between what is known nonverbally and what is to-be-learned verbally learning about counting. Therefore, knowledge about nonverbal counting principles can bridge learning about the verbal counting principles. Similarly, there is nothing in the nonverbal or verbal counting and arithmetic processes underlying children's arithmetic knowledge of the natural numbers that is inconsistent with the idea that every count number has a successor. Children should be able to learn with relative ease from novel inputs about the successor principle.

The mathematical principles underlying the numberhood of fractions are *not* consistent with the principles of counting and the child's idea that numbers are rendered when sets of things are counted and addition involves "putting together" two sets. One cannot count things to generate a fraction. Formally, a fraction is defined as the division of one cardinal number by another: this definition solves the problem that there is a lack of closure of the integers under division. To complicate matters, some counting number principles do not apply to fractions. Rational numbers do not have unique successors; there is an infinite number of numbers between any two rational numbers. One cannot use counting based algorithms for ordering fractions, for example, $\frac{1}{4}$ is *not* more than $\frac{1}{2}$. Neither the nonverbal nor the verbal counting principles map to a tripartite symbolic representations of fractions — two cardinal numbers *X* and *Y* separated by a line — whereas the formal definition of a fraction does. Related mapping problems have been noted by others (e.g., Behr, Harel, Post & Lesh, 1992; Fischbein, Deri, Nello & Marino, 1985; Greer, 1992). Therefore, if children bring to their early school mathematics lessons their theory of numbers that is grounded in counting principles and related rules of addition and subtraction, their constructivist tendencies could lead them to distort fraction inputs to fit their counting based number theory. Early knowledge of numbers might therefore serve as a barrier to learning about fractions.

There is some support for our proposals about early school mathematics learning. There are repeated reports of High School students having misunderstandings about fractions, especially as regards the idea that increasing values of the denominator maps to increasing quantitative values (e.g., Carpenter, Lindquiist, Brown, Kouba, Silver & Swafford, 1988; Behr *et al.*, 1992). In contrast, young school-aged children benefit from interviews related to the successor principle (e.g., Evans, 1983; Evans & Gelman, 1982; Falk, Gassner, Ben-Zoor & Ben-Simon, 1986). For example, Evans asked whether Kindergarten through Grade 3 children (5 to 10-year-olds) would benefit from suitable hints about the successor principle for natural numbers. Questions about the possibility of always adding to a given number were embedded in an interview about whether there is a largest number. Answers were never provided; yet, 50 and 85% of the Grade 2 and Grade 3 children, respectively, ended up telling us that there is no largest number and why. Even Kindergarten and Grade 1 children were able to give many relevant answers to the interview questions. We think that the even older subjects in Langford's (1974) study did not do as well as Evans' because Langford used a fill-in-the blank questionnaire. Our interview offered children opportunities to think about the relevant inputs for learning, a learning condition that can support induction of a mathematical principle (cf., Kitcher, 1982), that can engage Piaget's mechanism of reflecting abstraction (Piaget, 1980).

The Studies

The Successor Principle

Design Considerations

The English count list lacks a transparent decade rule for at least the first 40, and probably, first 130 count words. Although the rate at which English- and Chinese-speaking children master the first nine count words (Miller, Smith, Zhu & Zhang, 1995), Chinese speakers benefit from the fact that their language has a more transparent base-10 count rule for generating subsequent count words. (Miller *et al.*, 1995; Miller & Stigler, 1988). Limits on children's ability to generate larger and larger *N*s probably influences their performance on a successor task that asks them to think of larger and larger *N*s. Evans (1983) found a correlation between children's ideas about what is a large number, the pattern of their answers to addition questions, and their understanding that there is no largest number. Evans' assessment of verbal counting skills was indirect; scores were based on Successor Interview answers about large numbers. In the two studies reported here, children participated in tasks on counting and addition abilities designed specifically for these purposes.

Kindergarten (K), Grade 1 and Grade 2 children participated in two multi-phase, multi-day Successor Principle studies. For Study I, Day 1 tasks included the Rote-probed Counting and the Successor Interview. The order of the Counting and Successor tasks was counterbalanced, with half the children doing counting before the Successor Interview and half doing the Sucessor interview to start. Day 2 tasks included the Successor Interview Challenge and Addition items, in that order. Participants in Successor Study II always started with a Rote-probed Counting pretest on Day 1. Day 2 consisted of a Counting Item. Then Training experience (see below), the Successor Interview and the Addition Tasks, all in this order, were carried out over 1–2 more days. Successor Study I used our first variant of the Rote-probed Counting task to determine how high a child could "rote" count. Successor Study II used the second variant of the "Rote-probed" Counting interview. Pretest rote counting scores from Study II were used to match children (within each grade) for random assignment to one of three training conditions: Count Across Decades, Count Within Decades and Control.

The Training phase of Successor Study II was to evaluate what counting experience(s) are relevant to successor performance, knowledge of largish numbers in general, number sequences within decades beyond the child's counting range, or number sequences that crossed decade boundaries beyond the child's counting range. All training conditions started with a repeat of some of the Day Rote-probed Counting items. Across- and Within-Decades subjects then "helped puppets count" either across or within decades, respectively. Then they answered questions about numbers beyond their knowledge before starting the Successor Interview. Control children went from the rote counting trials to the remaining sets of items, the Successor Interview, challenges to the idea that there is a largest number and Addition tasks.

Subjects

Children in Study I attended a racially integrated school that served children from both its predominantly upper middle class Philadelphia neighborhood and those who voluntarily traveled to the school from one or more African-American, Puerto Rican and Asian communities in Philadelphia. The Study II school served a rural Pennsylvania university community that included lower-middle class and upper-middle class families. The rates of return on signed permission from a parent(s) for a child's participation were about 50 and 85% from School 1 and School 2, respectively. This included permissions for 15 children at School 2 who were older than typical for their grade. Therefore, data for children in K, Grade 1 and Grade 2 whose age was greater than (years:months) 6:6, 7:6 and 8:6 are not included in the analyses reported here. Data from one set of language disordered Kindergarten children were also excluded. The remaining *N*s and mean ages for the children in K, Grade 1 and Grade 2 in Study I were 52 (mean age 5:9), 27 (mean age 6:10) and 31 (mean age 7:10), respectively. Study II analyses are based on 36 K (mean age 5:11), 29 Grade 1 (mean age 6:11) and 26 Grade 2 (7:10).

The Tasks

Rote-probed counting procedures. The Rote-probed Counting procedures assessed a child's knowledge of the English numerlog counting algorithm. To start, a child counted as high as she could. Children who counted to 125 were stopped there; those who could not were asked "What comes after *N*?" and, if necessary, the experimenter said "($N - 2$), ($N - 1$), *N*, ..." This part of the Rote-probed Counting session ended if subjects could not count further even after these prompts. Those children who could count to 125, either spontaneously or with the latter probes, were asked to count on from 190. If successful, they were stopped at 222, then asked to count on from and 490 stopped at 522, then asked to count on from 995 and stopped at 1012. Children who had difficulty reaching 200 were encouraged to teach a series of puppets to count on. The numbers counted by each puppet depended upon the child's rote count or response to the preceding puppet (see Table 1). The puppet gave subjects a running start and the experimenter asked the child to tell the puppet what "comes next".

Study II counting training experiences. Within and across training children were asked to help a puppet count by trying to tell the puppet the next number whenever it got stuck. They did this for six different puppets, each of which focused on a different counting boundary: 99–100, 109–110, 199–200, 499–500, 999–1000 and 1009–1010. The puppets' counts were fixed and not adapted to subjects' rote count. Within-boundary children were told that sometimes the puppets liked to count to themselves, which allowed us to keep children from hearing the transition from one decade or hundred to the next. The experimenter gently reprimanded the puppets for counting to themselves just in time for its second sequence to be audible. For example, the first Within puppet counted from 94 to 98. Prompted with 99 by the child, the puppet then "continued counting" quietly for 100 and 101 but out loud for 102–106 so that the children only heard the puppet count from 102 to 106. Across condition trials involved puppets counting decade boundaries before the puppets "got stuck". For example, Puppet 1 counted from 99 to 103 and then, to make sure it remembered the child's prompt, backed up to 97 and tried again, only to get stuck

Table 1
Procedures for Determining Puppets' Counts in the Probed Counting Task

Speaker	"Instructions" for speaker
1. Experimenter	Determine highest number child reached in rote count. If rote count < 125, go to (2); if rote count = 125, go to (13)
2. Experimenter	You can count pretty high. You'll certainly be a big help. You see. I have some puppets who are just learning to count. They're going to count for you, but when they get stuck, you help them out, okay?
3. Puppet 1	Counts from 1 to subject's highest number minus 3 (e.g., if rote count to 29, puppet counts to 26)
4. Puppet 2	Counts from 1 to X-ty 2 where X = first decade higher than child counted after Puppet 1 (e.g., if child stopped counting on at 29 with Puppet 1, Puppet 2 counted to 26).
5. Puppet 3	Counts from 1 to X-ty 8 where X = same decade as child conter after Puppet 1, e.g., 28. If $X < 100$, go to (6); if $X = 100$ go to (9)
6. Puppet 4	Counts from 1 to X-ty 4, where X = two or three decades higher than subject counted after Puppet 3 but < 100 (e.g., 54). Avoid 44 because of acoustic confusions.
7. Puppet 5	Counts from 85 to 103
8. Puppet 6	Counts from 100 to 112. Stop
9. Puppet 4	Counts from 140 to 158
10. Puppet 5	Counts from 180 to 197. If count < 200, go to (11). If count $X = 200$, go to (12)
11. Puppet 6	Counts from 210 to 223. Stop
12. Puppet 6	Counts from 990 to 999. Stop
13. Experimenter: I can tell you can count really high	Try going on from 190. If necessary, give a running start: "190, 191, 192". If child stops at 199, "200 is the next one. Start at 195 and counter from there": If child still cannot go beyond 200, go back to (3).
14. Experimenter: stop subject at 222	Try going on from 490. (Child may need prompt at 500.)
15. Experimenter: stop subject at 522	Start at 995 and count from there. (Child may need prompt at 1000.)
16. Stop at 1012	End

at 101. Each of the six puppets counted twice, each time reciting five numbers before getting stuck (see Table 2). The Within and Between procedures were designed to make sure every subject in a condition at least *heard* the planned input and had the same experiences. Therefore, they were not asked to count on as high as they could themselves but only asked to give the puppet the next number. If a child was unable to prompt a puppet when it got stuck, the experimenter asked the child, "Do you think X (the subject's number) would come right after N, or do you think it would be $N + 1$?". Then the planned probes continued with the child helping the next puppet so it could have its turn.

The Successor Interview. The interview embeds the principle that one can always add to any natural number N and get $N + X$. The Study I Successor Interview, like Evans' (1983; Evans & Gelman, 1982), had a flexible format. When scoring these answers, we found some questions were ambiguous or led to digressions. For example, Study I Kindergartners might answer, "Is there a biggest number?", as if we had asked, "Is there a big

Table 2
Count Sequences Recited by Each Puppet in the Experimental Probed Counting Tasks Used in Successor Study II

	Experimental condition	
	Across decades	Within decades
Puppet 1	99–103 97–101	94–98 102–106
Puppet 2	108–112 109–113	103–107 114–118
Puppet 3	198–202 197–201	193–197 202–206
Puppet 4	497–501 499–503	492–496 504–508
Puppet 5	997–1001 998–1002	992–996 1003–1007
Puppet 6	1009–1013 1008–1012	1004–1008 1013–1017

(or bigger) number?" The more streamlined and focused interview of Study II required experimenters to include all planned target questions and resist introducing new questions, although they could backtrack if a child was confused or seemed to be changing his/her mind. The streamlined Study II version was easier to administer and score; still, children were no more or less successful. Therefore, we focus on the Study II version below (Hartnett, 1991 provides further details about the Study I interview.)

To start, the experimenter asked, "Do you ever just think about numbers, like when you're lying in bed or walking outside or something?" She then asked the child to think of a "really, really big number" and add to it three times — first two ($N + 2$), then four (($N + 2$ answer) + 4), then one (N($N + 2 + 4$ answer) + 1). Children were not expected to produce exact sums; instead, the goal after each addition was to ask if the child knew that their big number still got bigger when another was added to it. If a child did not say this, the experimenter asked, "Is the number you get now the same as the number you picked as your big number, or is it bigger, or is it smaller?" If necessary, the experimenter demonstrated adding two or four objects to a set of three objects to make the point that adding results in a bigger number.

The interview said "Can people always add to make a bigger number, or is there a number so big, we couldn't make it any bigger?" If children said people cannot keep adding, they were asked why not. If they still insisted on stopping, they were asked what would happen if a person tried to add anyway. If the addition was allowed, children were asked again if people can always add. The other target questions are shown in Table 3. Some target questions were repeated. Once a child said people can always add, or had been through the entire series of questions about adding, the focus of the interview was shifted to counting as a way of generating larger values of N. "If we count and count and count, will we ever get to the end of the numbers?" If the child said, "No", she was asked why not. The experimenter then asked, "What if we cheat, and instead of starting at one, we start counting from a really high number? Then could we get to the end of the numbers?" If the child still said a person could not count to the end, she was asked if there

Table 3
Successor Interview Target Questions for the Order Effect Study

Can people always add to make a bigger number, or is there a number so big we couldn't make it any bigger?
If we count and count, will we ever get to the end of the numbers?
Is there an end to the numbers?
What if we cheat and instead of starting at one, we start counting from a really high number? Then could we get to the end of the numbers?
Is there a last number?
(Googol challenge) ...So, if someone came up to you and said, "I think a googol is the last number there is", would you believe him?
(Puppet 1 challenge) ...Is that right? Do you think we have to stop adding at quintillion 100?
(Puppet 2 challenge) ...Could that be right? Do you think if he counts long enough he'll get to the end of the numbers?

is a last number. If she said there is no last number, she was asked whether she would believe someone who thought googol is the last number, and the experimenter tried to elicit a proof. Children were told that a Googol is a "really big number" that "comes after the hundreds and thousands and millions and billions".

If a child said that a person could count to the end of the numbers, he was asked, "Is there an end to the numbers?" and, if appropriate, "How will we know when we get to the end of the numbers?". If the subject still insisted on there being an end, he was asked if there is a last number and, if so, what the last number is. As a final check, the experimenter asked, "Are there more numbers after that, or is *N* the last? How do you know?". If the subject said there are more numbers, the contradiction was pointed out and the experimenter went on to the googol challenge. The googol challenge was omitted, however, if the subject insisted there is a last number and there are no more numbers after it.

Follow-up challenges. These were presented by puppets. One said, "I've thought of a really big number that's so big, it's almost at the end of the numbers. Do you know what it is? It's quintillion 99". The experimenter then asked the puppet to add one to its number, and the puppet responded, "It's even bigger! Now I'm up to quintillion 100". When asked to add one again, however, the puppet refused, saying, "I can't. Quintillion 100 is the last number. We have to stop adding". The subject was asked if she agreed and if so, "What would happen if we tried to add anyway?" If the child disagreed, she was asked, "Do you think we could always add, or is there a last number where we have to stop?". The second puppet appeared and said, "I've been counting and counting and counting, and I know I'm going to get to the end of the numbers soon. I'll count the rest of the day and all night and then I'll finish. I'll get to the end of the numbers". The experimenter asked the child, "Could that be right? Do you think if he counts long enough he'll get to the end of the numbers? Why is that?".

Addition. The addition tasks determined whether or not subjects could add systematically, i.e., give successively sums larger than either addend and do so in ascending order. In Study I, children were asked for a "really big number" and then asked to add one to it three times. They were also asked to add one repeatedly to 101 and 1200, e.g., "What would you get if you added one to 101? And if you added one to that? And if you added one to that?". In Study II, the task was expanded. Children were also asked to: (a) add each of 1 and 100 three successive times to 200; and each of 1 and 2 to 101. First and second grade children were also asked to add Ns of 1 and 1000 to 4000 and do so for three times in a row. Pairs of test numbers that shared an addend were given in order to determine whether children would know that different problems must result in different answers. If, at any point, a child was unable to give a sum, he or she was asked if adding results in the same number, a bigger number, or a smaller number. The experimenter occasionally gave the first sum and asked the child to add one to that number three times.

Scoring Study II

*Rote-probed counting**. By design, the maximum rote count was 125. Children's rote counts were scored at three levels of stringency: (1) *Perfect:* the number counted without any errors, allowing only for spontaneous self-corrections. (2) *Rote Counts with Allowable Errors:* in this case, omissions, repetitions, intrusions, or reversals were allowed if they were followed by an errorless string at least twice as long as the error string. The score was the number reached before the child started to make nonallowable errors. For example, a child who counted "1...15, 18, 19...29", was credited with counting to 29. (3) *Best Rote Count:* this was the highest number a child reached, even if that number was not reached until a child responded to the first puppet during the probe part of the rote counting task. For example, if a child counted correctly to 29 and in response to the first puppet's count from 1 to 26, continued on to 34, her Best Rote Count score would be 34.

A child's response to the puppets' and/or experimenter's requests to count in ranges higher than the child was scored as: (1) *Not correct*: children gave the wrong answer or repeated a number said by the puppet. (2) *Partial decade*: the child's follow through was correct but it did to complete the decade. (3) *Completed decade*: a subject counted on to the end of the decade. (4) *Crossed decade*: a child counted on to the end of the given decade and then at least one number in the next decade. For the latter two categories, errors were allowed if they were followed by an errorless string as least as long as the error string and as long as the child met the relevant criteria for the category. Counting on from 97 to 100 was classified as Completed decade as opposed to Crossed decade, for those children whose rote counts stopped at 100.

Since we were especially concerned with the child's knowledge about generating count terms, all analyses of rote counting ability reported here are based on Best Rote Count scores.

The Successor Interview. Children's Y/N responses to the interview target questions listed in Table 3 and the explanations they gave — either spontaneously or following the experimenter's probes — to justify their Y/N answer, were scored separated. Then, a pair of time-graphs — one for the Y/N pattern of answers over trials and one for the explanation kinds over trials — were constructed for each child. The children's protocols were classified into one of three levels of understanding the Successor Principle: Understanders, Waverers and Nonunderstanders.

Yes–no responses and explanations to target questions. Each individuals' Y/N response to the interview target questions was coded as 0, 1 or 2. Correct responses were coded as 2; incorrect responses as 0; and unscorable responses (e.g., "I don't know") as 1. The database for explanations excluded all "Yes", "No" or "I don't know", or verbatim

*See Hartnett (1991) for more details of the Rote-probed Counting schemes.

repetitions of the question. Additionally, if answers in Study I followed a leading question, these too were dropped. A first pass through the explanations yielded the 28 explanation types in Hartnett (1991). These were then classified as: Infinite, Ambiguous, Unacceptable explanations, or Finite explanation. Nominal scores were assigned to each of these explanation kinds, with Infinite ones scored as 2; Ambiguous and Unacceptable explanations, being neither principally right nor wrong, 1; and Finite ones as 0. The criteria for the four kinds of explanations were as follows: (a) *Infinite*: these had to include assertions that there is no biggest or last number. For example, "Because if you keep adding one to a number, you can keep getting a higher number. And you can keep adding one forever; you'll never get to the end". (b) *Ambiguous*: these had both a Finite and an Infinite explanation and thus were contradictory. For example, "There is (a biggest number). It's hard to tell which number it is because it's hard to count. Numbers go on forever, so it's hard to count. It's hard to tell"; "Infinity is the biggest number because that goes forever". (c) *Unacceptable*: these often were irrelevant, "Because we need sleep" or "Because the bigger it is the better it is". (d) *Finite*: these contained assertions that there is no biggest or last number. For example "There's no other one... It's like the alphabet. Z's the last letter in the alphabet".

All explanations from both studies were coded by the first author. A randomly selected sample of 175 and 177 explanations, from Study I and Study II, respectively, were coded by another independent rater. Overall, interrater agreement was 83%. Setting aside disagreements due to lapses in attention on the part of the second coder, interrater agreement for Study II was 92%, with full agreement on explanations coded as Infinite.

Time graphs. To prepare the data for the creation of time graphs for each child, interviews were divided into first and second halves on the basis of the number of Y/N responses subjects gave to target questions. For example, if a subject responded to 10 target questions throughout the course of the Successor Interview, the sixth through tenth responses were considered the second half. Then, *any* explanations that were given after the sixth Y/N response were considered second half explanations. Since the halfway point for the explanation data was yoked to the number of Y/N responses, it is possible for the number of explanations in the first and second half of the interview to differ.

Classifying time graphs. The classification of each child's graphs was done in two steps. First, each subject's pattern of Y/N responses to target questions was scored as being that of an Understander, a Nonunderstander, or a Waverer. Children classified as an Understander on the basis of their Y/N answer patterns, either (a) answered all target Y/N questions correctly, or (b) gave at least twice as many correct as unscorable response (e.g., "I don't know") throughout the interview *and* answer all second half Y/N questions correctly. A subject was scored as a Nonunderstander if he never answered a target Y/N question correctly in the second half of the interview. Subjects who were neither consistently wrong nor right were scored Waverers. These subjects' graphs contain a mixture of correct and incorrect Y/N responses to target questions throughout the interview.

Criteria for generating time graphs for explanation data were built on the kind of Y/N time graph already in hand for a subject. No explanation time graphs were made for Y/N Waverers. Since no pattern of explanation data could result in a change in their classification. To end up classified as Understanders or Nonunderstanders, children had to meet both the Y/N response criteria (described above) and the explanation criteria (described below). Those who only met the Y/N response graph requirements for Understanders, that is, those who failed to meet the explanation graph requirements for the same classification, were ultimately classified as Waverers.

Subjects who met the Y/N response graph requirements for Nonunderstanders were classified as Nonunderstanders if they did not give any Infinite explanations in the second half of the interview. Because Infinite explanations are inconsistent with the belief that there is a biggest number, the intrusion of any in the second half of the interview resulted in subjects being classified as Waverers instead. A similar consideration applied for the final classification of children as Understanders. Those who met the Y/N response graph requirements for Understanders were classified as Understanders if they did not give any Finite explanations in the second half of the interview. Because Finite explanations are inconsistent with the belief that there is no biggest number, the intrusion of any in the second half of the interview resulted in subjects being classified as Waverers instead.* Examples of time graphs for Waverers and Understanders are shown in Fig. 1 Fig. 2, respectively.

The addition task. Subjects' additions to 101 and 1200 were scored and analyzed in two ways. To be scored Correct on a problem, a subject had to give the correct sum to all three additions (e.g., 101 + 1 = 102 + 1 = 103 + 1 = 104). To be scored Ascending Order, a subject had to give three numbers in ascending order, all three of which were larger than the addends, for example. 1200 + 1 = 1300 + 1 = 1400 + 1 = 1500. A response such as 101 + 1 = 100 + 1 = 200 + 1 = 300 would not be scored Ascending Order because the first sum (100) is smaller than one of the addends (101). Nonstandard labels were allowed, as in, "Three thousand three hundred" plus one equals "Four hundred and four thousand". Concatenations, as in saying 101 + 1 equals "One hundred one, one" plus one more equals "One hundred one, one, one" were not allowed. Subjects who gave three correct sums following the experimenter's prompt on the first addition trial were scored Ascending Order on that problem.

Results and Discussion

Did children acquire the successor principle? Table 4 shows the percentage of children we classified (in each grade and each study) as Understanders, Waverers and Nonunderstanders. Note that relatively few of even the youngest subjects firmly rejected the Successor Principle (i.e., were Nonunderstanders). Rather, Kindergartners and First graders were most commonly classified as Waverers. That is they had some understanding of the concept in question, but were less likely than Second graders to answer consistent across question types. Even though we scored more stringently than Evans (1983; Evans & Gelman, 1982), we too found that the majority of Second graders understand that every natural number has a successor.

The results in Table 4 probably underestimate the extent to which younger children did understand because some of the Y/N target questions were problematic, especially in Study I. As shown in Fig. 3, Kindergartners and Grade 1 children had considerable difficulty

*Subjects' time-graphs were also scored according to three stricter sets of criteria (see Hartnett, 1991). While the leniency of the scoring procedure obviously affected the number of subjects classified as Understanders, the overall pattern of results was the same by all four scorings. That is, the distribution of Understanders across grades and counting abilities was the same for all four sets of criteria.

Time-graphs of Two Subjects Classified as Waverers with Respect to the Successor Principle

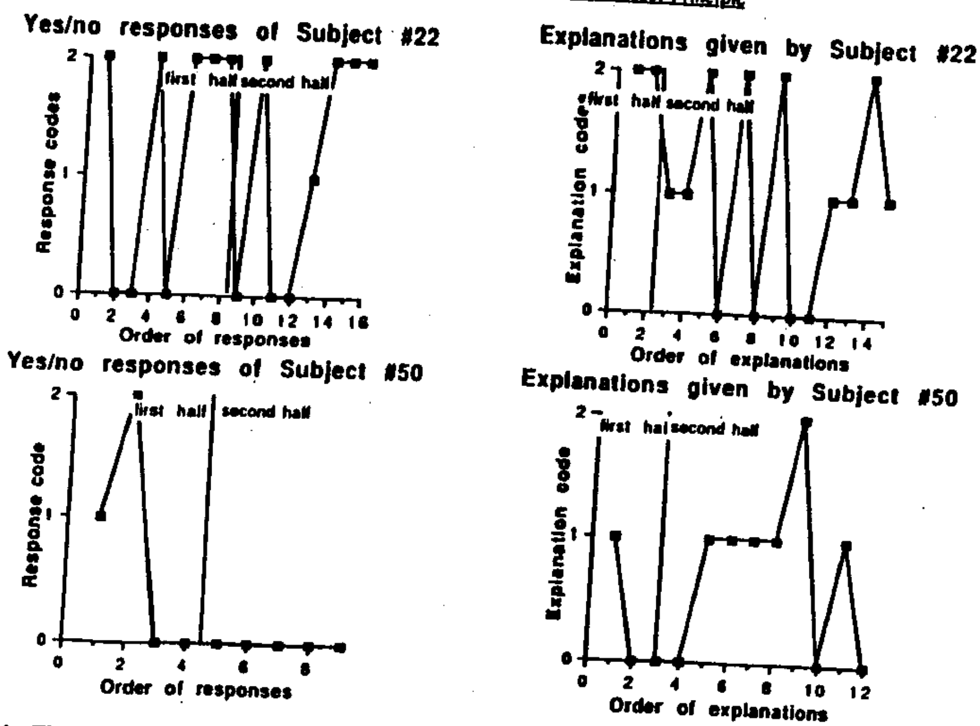

Figure 1. Time-graph for two subjects from Study I who were classified as Waverers on the Successor principle.

with "Is there a biggest number of all the numbers?" Similarly, "Is there a last number" was answered incorrectly more often than not by these two younger groups. In contrast, "If we count and count and count, will we ever get to the end of the numbers?" and "Can we always add one more, or is there a number so big we'd have to stop", was more likely to be answered correctly by all age groups. No question posed particular difficulty for Second graders.

Thus, the grade differences in Table 4 reflect, in part, the extent to which children could interpret correctly all of the target questions. Taking this into account, we conclude young children can understand and make use of the implications of repeated counting and addition. The following protocols illustrate this as well as the differences between Waverers and Understanders.

D.A. (5 years, 9 months; a Waverer)

> (D.A. has already allowed addition to his biggest number but insists a biggest number exists.) ...WOULD I EVER HAVE TO STOP ADDING ONE TO THAT NUMBER, OR COULD I JUST KEEP ON ADDING ONE? You could keep adding one. WELL, THEN, IS THERE A BIGGEST NUMBER, OR DO YOU THINK THE NUMBERS GO ON AND ON AND YOU CAN KEEP ADDING ONE? You can keep adding one. YOU CAN? IS THERE A BIGGEST NUMBER, THEN? Yeah. HOW CAN THERE BE A BIGGEST NUMBER IF YOU CAN KEEP ON ADDING ONE? Because it will be longer. ...DO YOU THINK THAT THERE IS A LAST NUMBER, OR CAN WE ALWAYS ADD ONE MORE TO MAKE IT BIGGER? You can add one more until it gets real big. DO YOU THINK YOU EVER HAVE

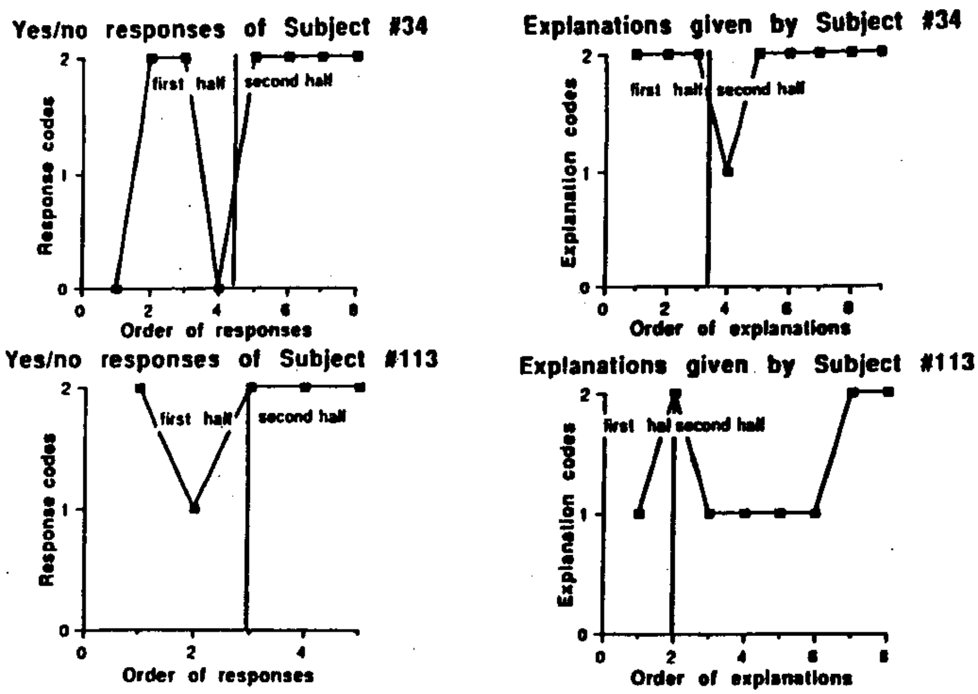

Figure 2. Time-graph for two subjects from Study I who were classified as Understanders on the Successor principle.

Table 4
Level of Understanding of the Successor Principle Across Grades: Percentage of Children Who Were Understanders, Waverers or Nonunderstanders Given Their Time-graphs Classification

	Percentage of children in each grade			
	Kindergarten	Grade 1[a]	Grade 2	All grades
Understanders				
Study I	27	26	52	37
Study II	25	52	58	43
Waverers				
Study I	50	63	39	50
Study II	53	34	38	41
Nonunderstanders				
Study I	23	11	10	16
Study II	22	14	4	14

[a]See Hartnett (1991) for a discussion of why the difference between the Grade 1 results for the two studies most probably reflects the inexplicable differences in permission rates for just this grade at the school for Study I.

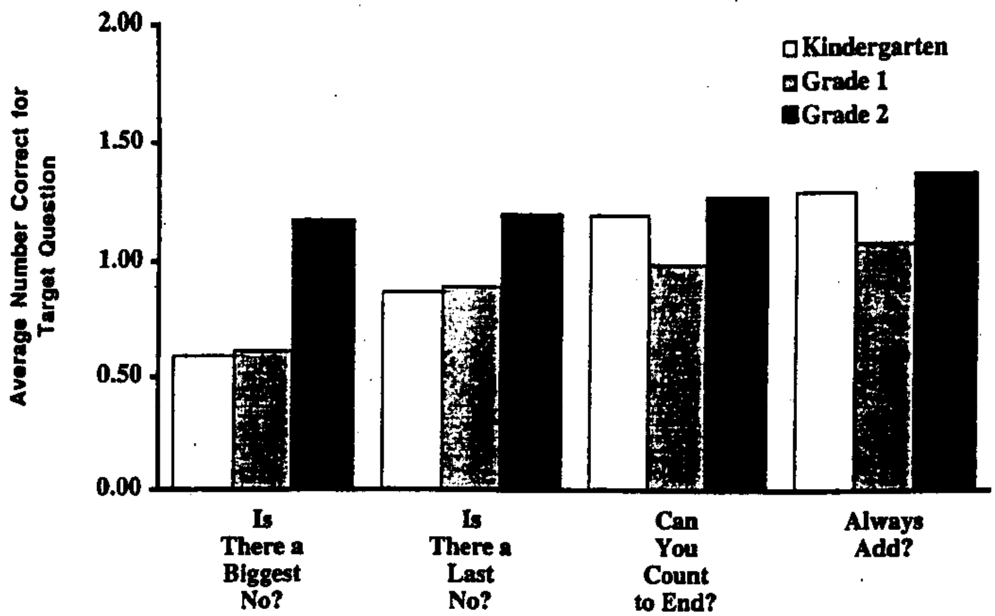

Figure 3. Difficulty of different Study I target questions as function of grade.

TO STOP ADDING ONE MORE? (shakes head no). NO? WHY NOT? Because there would be the highest number in the whole numbers. THERE WOULD BE A HIGHEST NUMBER? OR COULD YOU ADD ONE TO THAT AND MAKE IT EVEN HIGHER? You still put one and they get real higher.

A.R. (7 years, 3 months; an Understander)

...IF I THOUGHT OF A REALLY BIG NUMBER, COULD I ALWAYS ADD TO IT AND GET A BIGGER NUMBER? OR IS THERE A NUMBER SO BIG THAT I COULDN'T ADD ANYMORE; I WOULD HAVE TO STOP? You could always make it bigger and add numbers to it... IF I COUNT AND COUNT AND COUNT, WILL I EVER GET TO THE END OF THE NUMBERS? Uh uh. WHY NOT? Because there isn't one. THERE IS NO END TO THE NUMBERS? Uh uh. HOW CAN THERE BE NO END TO THE NUMBERS? Because you see people making up numbers. You can keep making them, and it would get higher and higher... OH. WELL, WHAT IF I CHEATED, AND INSTEAD OF COUNTING FROM ONE I COUNT FROM A REALLY HIGH NUMBER? COULD I GET TO THE END THEN? Uh uh. YOU MEAN THERE'S NO LAST NUMBER? Uh uh. Cause when you get to a really high number, you can just keep on making up letters and adding one to it. ...OKAY. WELL, WHAT IF SOMEBODY CAME UP TO YOU AND SAID THAT A GOOGOL WAS THE BIGGEST NUMBER THERE COULD EVER BE? DO YOU KNOW WHAT A GOOGOL IS? Uh uh... WHEN YOU MAKE A GOOGOL, YOU WRITE A ONE AND A HUNDRED ZEROES. IT'S A REALLY BIG NUMBER. SO, IF SOMEBODY CAME UP TO YOU AND SAID, "I THINK A GOOGOL IS THE BIGGEST NUMBER THERE COULD EVER BE", WOULD YOU BELIEVE HIM? Uh uh. Because you could put one and a thousand zeroes on it.

The following sample of a protocol from a Nonunderstander illustrates our conclusion that Waverers and Understanders are more similar than not regarding their understanding of the role of counting on and repeatedly adding.

K.B. (5 years, 8 months; a Nonunderstander).

...WHY DO WE HAVE TO STOP [COUNTING]? Cause you need to eat breakfast and dinner. YES. BUT THEN YOU COULD START IT UP AGAIN AFTER YOU ATE. You forget where you stopped.

...WHAT IF WE GOT TO THAT NUMBER (K.B.'S BIGGEST NUMBER), AND WE TRIED TO ADD ONE ANYWAY? WHAT WOULD HAPPEN? I guess you'll be old, very old... HOW WILL YOU KNOW WHEN YOU GET TO THE END? Well, you can stop any time you want to.

Overall, the above results demonstrate that young children either came to our interview knowing or ready to learn quickly about the Successor principle. To be sure, we did observe differences in children's ability to answer successor questions consistently. This difference in interpretative competence could reflect differences in understanding of the correct mathematical interpretation of phrases and words that are used in mathematical conversations like our Successor Interview. Evidence for this hypothesis comes from analyses of the relationship between children's understanding of verbal counting and adding and their performance on the Successor Interview. In what follows, we first review the relationship between verbal counting and adding knowledge as it relates to understanding of the successor principle. We then return to the proposal that ideas about the successor principle may become more and more mathematical.

Understanding the successor principle and knowledge of verbal counting and addition. The Rote-probed counting results for the two studies are very similar. There was no school effect on Best Rote Count scores for Kindergarten and First grade children in Study I, the percentage of trials on which the first response to the puppet's repeat of their count was correct was 85.1, 94.1 and 98 for Kindergartners, First Graders and Second graders, respectively. The respective figures for Study II, were 87.3, 94.6 and 99.5.

Table 4 shows the percentage of children in different grades whose best counts fell within the 1–20, 21–100 and 101–125 ranges. Interestingly, very few children were limited to counts between 1 and 20. Table 5 data for Study II Kindergartners shows them less likely to count over 100 than their grade-mates in Study I. However, Kindergartners at the Study II school were taught to count to 100 and no further, while no such limit was placed on the Kindergartners in Study I. This is surely why 10 of the 36 children in Study II stopped at 100 whereas only 2 of the 52 Study I Kindergartners did the same.

Verbal counting and successor principle knowledge. The ability to count and add on can, rather roughly, be characterized as necessary but not sufficient for learning about the successor principle. Children who have facility with verbal counting algorithms can gener-

Table 5
Summary of the Rote-counting Abilities of Children in Studies I and II: Percentage of Children in Each Grade Whose Best Rote Count Falls into the 1–20, 21–100 or 101–125 Range

	Study I			Study II		
Range of Best Rote Count	Kindergarten (52)[a]	Grade 1 (27)	Grade 2 (31)	Kindergarten (36)	Grade 1 (29)	Grade 2 (26)
1–20	2	4	0	2	0	0
21–100	48[b]	8	0	67[b]	10	0
101–125	50[b]	89	100	31[b]	90	100

[a]Number in brackets = N children per grade in each study. [b]Note that there is a difference between the distribution of Kindergarten children in each of the studies whose counts were in the 21–100 as opposed to the 101–125 count range. This is most likely an artefact of the different teaching methods at the two sites. Those in the second study were taught to count only to 100. No such restriction was placed on the Kindergartners in Study I. This probably explains why only 2 of the 52 children in Study I had a Best Rote Count of exactly 100. In contrast 10 of the 36 Kindergartners in Study II stopped at 100.

ate novel successive counting tags; similarly, counting on is a way to produce novel examples of natural number successors. Hints to add repeatedly, in combination with sufficient knowledge of how to generate count words, might lead a child to generate or recognize test cases related to the induction that there always is another natural number. Some relevant evidence comes from an interesting order effect in Study I.

The Study I order effect. A random half of the children in Study I were in the Counting tasks before the Successor Interview, the other half in the Successor Interview before the Counting tasks. No matter what the measure — Y/N scores, distribution of explanations, and classification as Understanders, Waverers, or Nonunderstanders — Count-first subjects performed better in the Successor Interview than Interview-first subjects (Fig. 4). If children were scored as Understanders in Study I ($N = 37$), the probability that they were in the Count-first order is 0.73. The difference in the number of Count-first and Interview-first subjects classified as Understanders is reliable for First and Second graders, and nearly so for Kindergartners (Kindergarten: $\chi^2 = 2.92$, $0.05 < p < 0.10$, $df = 2$; Grade 1: Fisher exact probability = 0.029; Grade 2: $\chi^2 = 3.89$, $p < 0.05$, $df = 2$). Fig. 5 shows the Understander, as well as the related NonUnderstander and Waverer results. Because there were so few Grade 1 and Grade 2 Nonunderstanders, the three grades were combined to show no effect of task order on the likelihood of being classified as a Nonunderstander ($\chi^2 = 0.36$, $p > 0.50$, $df = 2$). Of the 55 Waverers, 21 (38%) were Count-first subjects and 34 (62%) were Interview-first subjects. Only among the First graders were Interview-first subjects reliably more likely than Count-first subjects to be classified as Waverers (Fisher exact probability = 0.015).

It was not that subjects simply did better on whichever task they got second. Count-

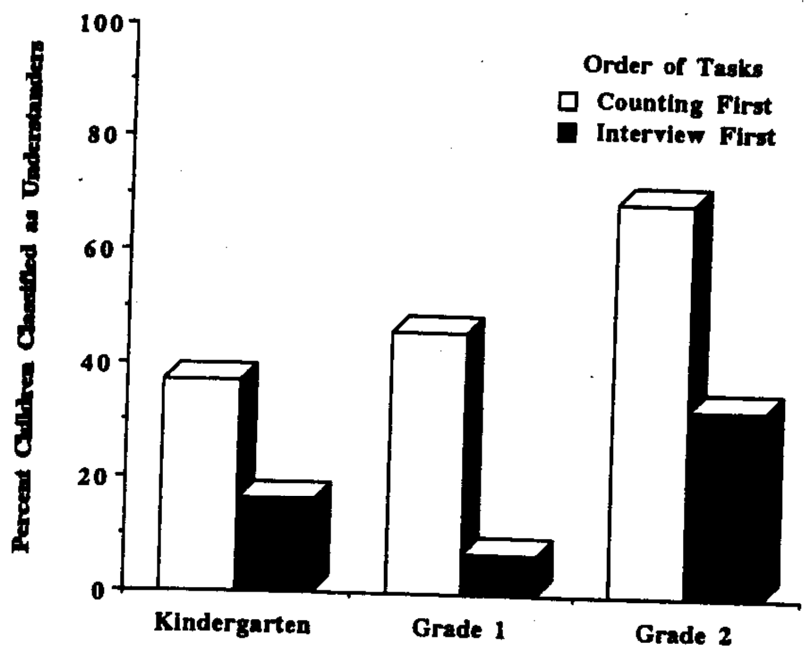

Figure 4. The effect of Study I task order on performance during the Successor interview.

first subjects were no better at counting than were Interview-first subjects. An analysis of variance of Kindergartners' and First graders' errors revealed a main effect of grade ($F_{1,75} = 9.96$, $p < 0.01$), but no effect of task order ($F_{1,75} = 0.66$, $p > 0.40$) and no interaction ($F_{1,75} = 0.25$, $p > 0.60$). Second graders were not included in this analysis of variance because their rote counts clearly were cut off within their range of competence. The two order conditions were also compared, grade by grade, on five other variables that could conceivably affect performance in the Successor Interview: (1) mean age, (2) classroom teacher, (3) length of protocol, (4) experimenter conducting the interview, and (5) date interviewed (i.e., whether subjects were among the first or the last interviewed during the course of the study). There were no reliable differences between the two conditions on any of these variables (Hartnett, 1991).

How might the Counting tasks in Study I have helped children on the Successor Interview? One possibility is that they simply primed children for the Successor Interview, for talk about generating numbers. A second possibility is that the critical experience for the Count-first subjects was hearing the puppets or experimenter say numbers beyond their own rote count range. Once in the interview they may have reasoned, "If there are numbers higher than the ones I know, I guess there could be numbers higher than googol, too". A third possibility is that Count-first subjects were more likely to induce the Successor principle because they learned both that large numbers exist, and something about how to generate those large numbers. The Rote-probed counting task taught at least some children to count in higher ranges than they could before. One Grade 1 child who was told that 200 follows 199 and 500 follows 499, exclaimed, "500? Oh, now I get it! You go one number higher". He was then counted on from 695 into the 700s without help. Study II was designed to separate these three possibilities. Children were assigned to one of three conditions that differed in the type of counting experience they received before the Successor Interview. Those in the Control condition were given the standard Rote-probed Counting experience; those in the Across condition started the same way and then heard puppets count across decade or hundred boundaries, such as 99, 100 and 499, 500, and after hearing the Rote-probed sequence, those in the Within condition heard puppets count in the same ranges as Across puppets did, but the puppets did cross from one decade or hundred to the next. For example, Puppet 4 counted from 492 to 496 and then 504 to 508.

Interview-first subjects in Study I had no focused counting experience until after the Successor Interview. If the function of the counting experience for the Count-first children in Study I was to prime them for the Successor Interview, then we should expect children to do comparably well in the three Training conditions. All had the opportunity to count before the interview began. If, however, the relevant experience for Count-first subjects in Study I was an opportunity to hear numbers larger than they could count themselves, the children in the Across-decade and Within-decade conditions of Study II should outperform the Control children on the Successor Interview. Finally, if the relevant aspects of the counting experience in Study I included an opportunity to hear very large numbers and learn something about generating these numbers, children in the Study II Across-decade condition should be classified as Understanders of the Successor principle more often than either Within and Control children.

In fact, there was no effect of experimental condition on performance level in the Successor Interview. Control subjects, who simply had by rote-probed counting experience before beginning the Successor Interview, were just as likely to do well in the interview as the Across and Within subjects, who heard puppets count into the hundreds and thousands. Table 6 shows, for each grade, the number and percentage of subjects in each condition in Study II who were classified as Understanders. The first grade Control subjects' somewhat better performance for the grade is not reliable (Fisher exact probability = 0.07). Table 6 also presents the number and percentage of subjects in each condition who were classified as Waverers and Nonunderstanders. Once again, there was no effect of condition.

It is not that all of the Study I children who had the Counting assessment before the Successor Interview did better. As expected, children most likely to benefit from the counting experience in Study I had some knowledge of the count sequence beyond 100 and understood that repeated addition increases the value of N.

Ideas about big numbers. Recall that in Study I the Successor Interview began with an invitation to the child to tell the experimenter a "really, really big number". (Study II asked children to think of one in their head.) Across all subjects, be they in the Count-first or Interview-first condition, the size of the numbers named at the start of the Successor Interview was related to performance level in the interview. Children who gave numbers larger than 100 as their initial big number were reliably more likely to be classified as Understanders than children who gave smaller numbers. Eighty-seven percent of all Count-first subjects named numbers larger than 100; in contrast, only 48% of the Interview-first subjects started with numbers at least this large. This difference was reliable for Kindergartners ($\chi^2 = 12.24$, $p < 0.001$, $df = 1$) and Second graders (Fisher exact probability = 0.021). Therefore, one effect of the Count-first condition was to increase the value of children's first choices of a large number.

Evans (1983) found that the size of subjects' ideas of a big number correlated with their performance in the interview. In particular, all but one of Evans' subjects who gave numbers smaller than 100 at the start of the interview failed to make the induction that addition can continue indefinitely. Because these subjects were also less likely than those who named numbers larger than 100 to add systematically and conserve, Evans identified 100 as a transition point for number knowledge. A similar result holds for our study. Children who gave numbers larger than 100 as their initial big number were reliably more

Table 6
Percentage of Children in Each Experimental Condition Who Were Classified as Understanders, Waverers or Nonunderstanders in Successor Study II

Condition and group	Kindergarten	Grade 1	Grade 2	All grades
Understanders				
Across	36	40	56	43
Within	25	40	67	42
Control	15	78	50	43
Waverers				
Across	36	40	44	40
Within	58	40	33	45
Control	61	22	38	43
Nonunderstanders				
Across	27	20	0	17
Within	17	20	0	13
Control	23	0	12	13

likely to be classified as Understanders than children who named smaller numbers (χ^2 = 8.78, $df = 2$, $p < 0.02$; partitioning the degrees of freedom shows the reliable difference to be between Understanders and the other two classifications). Specifically, of the 35 subjects who gave numbers less than or equal to 100, 5 were classified as Understanders, 22 were classified as Waverers, and 8 were classified as Nonunderstanders. Of the 72 subjects who gave numbers larger than 100, 31 were classified as Understanders, 31 were classified as Waverers, and 10 were classified as Nonunderstanders.* In short, 43% of the subjects whose initial big number was greater than 100 were classified as Understanders. Only 14% of the children who named smaller numbers did as well in the interview.

Figure 5 shows that most Nonunderstanders confined their answers to the hundreds or smaller numbers when naming a really big number. Some used nonstandard nomenclature to give the largest possible number within the limitations of their vocabulary. For example, one First grade Nonunderstander offered "Ninety-nine hundred and eighty-seven". Waverers were most likely to give relatively small numbers or numbers in the thousands. The modal response for Understanders, however, was to give an "illion" number, such as million or billion. Although subjects classified as Understanders tended to be good counters, not all good counters were classified as Understanders. Knowledge of the term *infinity* did not ensure success in the interview. Of the 31 Study I subjects using this term, 10 (32%) were classified as Understanders, 15 (48%) as Waverers, and 6 (19%) as Nonunderstanders.

Knowledge of counting and addition. Being able to count past 100 is one condition for the order effect. Only those Kindergartners who could count beyond 100 benefited from the counting experience (see Fig. 6). There were more Count-first than Interview Kindergartners who could count beyond 100 who passed the interview (Fisher exact probability = 0.013). A second condition for the order is skill at repeated addition (Fig. 7). Those Kindergartners who were able to add with some success were reliably more likely to benefit from the counting experience than were those who could not add at all (Fisher exact probability = 0.02).

Although we used subjects' Day 1 rote counting ability to match the three experimental conditions in Study II, perhaps there were other differences; rote counts were not a reliable measure of counting knowledge. An analysis of variance of Day 2 Rote Count scores revealed the expected main effect of grade ($F_{2,\,58} = 19.33$, $p < 0.001$). Kindergartners and First graders showed no differences among the three experimental conditions ($F_{2,\,58} = 1.13$, $p > 0.30$) and no grade by condition ($F_{1,\,58} = 2.19$, $p > 0.10$). (Second graders were not included in the latter analysis because all but two reached 125.) Similarly, there were no differences in addition scores between the experimental conditions. A grade by condition analysis of variance on percentage correct showed a main effect of grade ($F_{1,\,59} = 16.44$, $p < 0.001$), but no effect of experimental condition ($F_{2,\,59} = 1.75$, $p > 0.15$) and no interaction ($F_{2,\,59} = 1.31$, $p > 0.25$). The First graders were better at addition than were the Kindergartners, but there were no differences among the three experimental conditions. (Again, Second graders were excluded since they were at ceiling on our measures.)

In sum, analyses of counting and addition knowledge suggest that there are two con-

*Note that the numbers total 107, not 110. Two subjects (one Understander and one Waverer) were not asked to give "a really big number". One additional Waverer responded with a nonsense number.

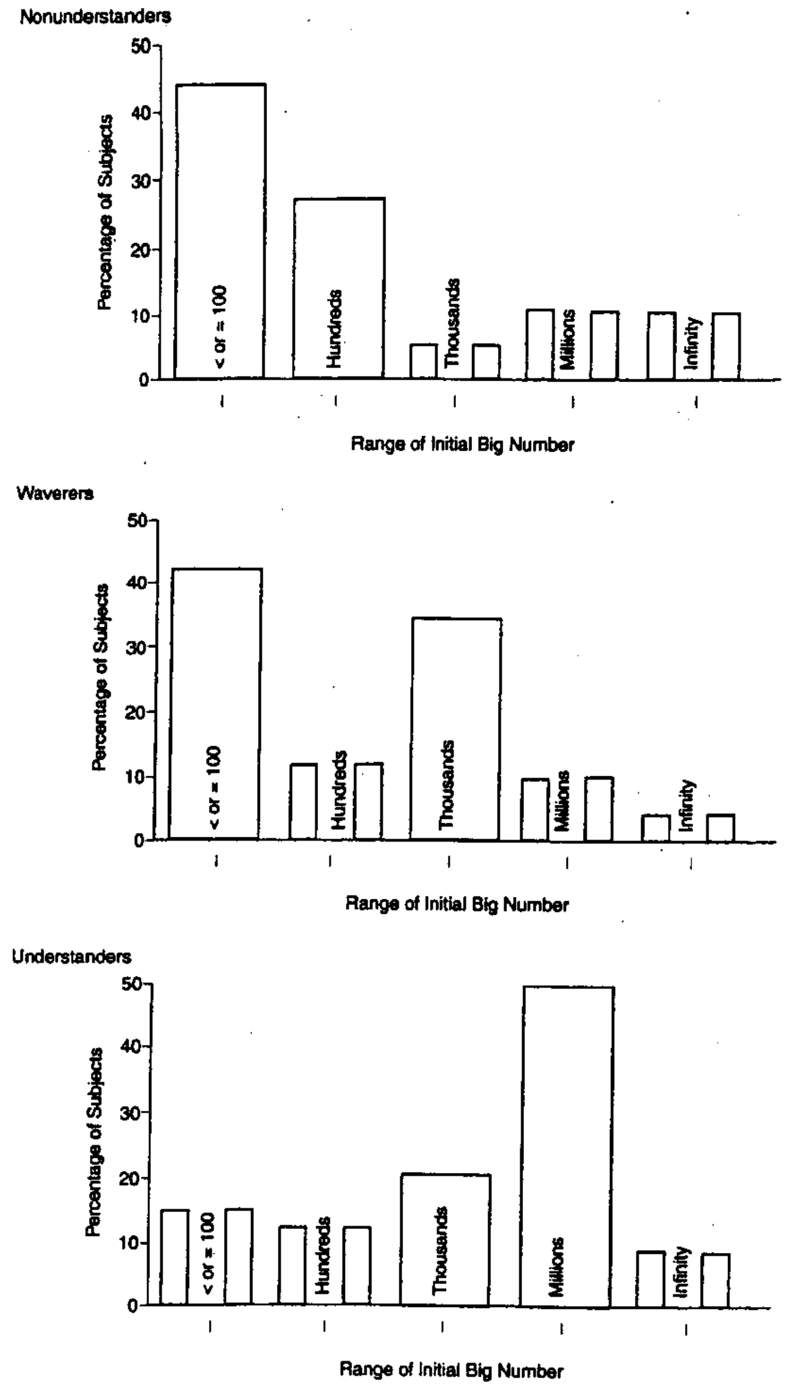

Figure 5. The relationship between a child's initial big number and Successor interview classification (Study I).

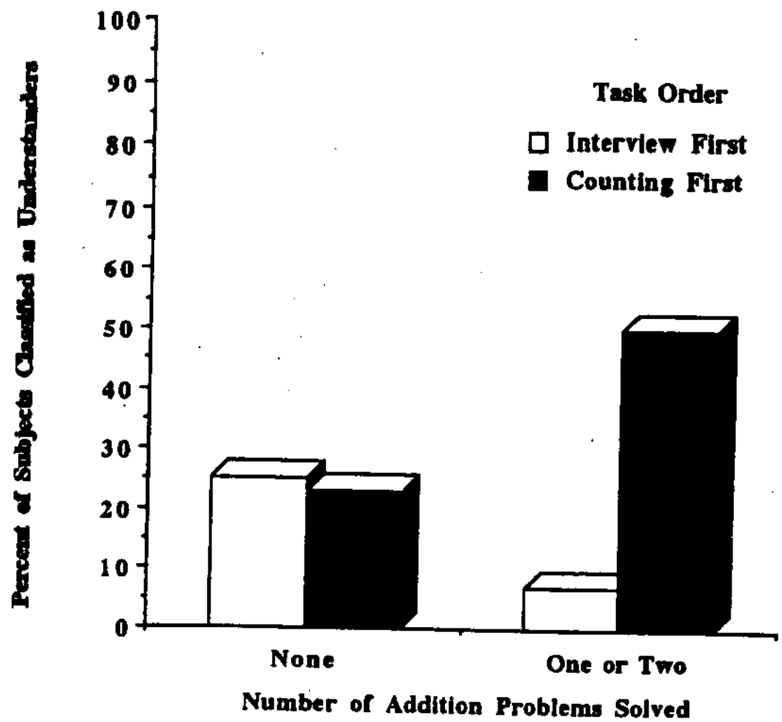

Figure 6. The effect of task order in Study I as a function of the addition skill of Kindergartners.

ditions for the Study I order effect: rote knowledge of the count sequence beyond 100 and understanding that repeated addition increases the value of *N*. The results of our two Successor studies converge on a common conclusion: beginning school-aged children can induce the successor principle for the natural numbers. Moreover, English speaking children who can count beyond 100 and generate increasing *N*s as a function of addition are more likely to benefit from a conversation that offers clues about the principle. Why? Once children master the sequence from 1 to 20 and the list of decade words, they have most but not all of the vocabulary they need to apply the recursive procedures by which larger and larger numbers are generated. As they count beyond 100, they come to learn that not only the digits, but also the decade terms, are recycled over and over. Children are still at work memorizing the teens and decade terms and are less able to appreciate that the count sequence is systematic, even when they hear others counting in high ranges. Those who can count beyond 100 are in a good position to realize the count sequence can be extended indefinitely. Given this, knowledge of the verbal counting algorithm can foster acquisition of the successor principle in at least two ways. It can provide children with a way to generate new examples and it can support their ability to get evidence that repeated addition also increases *N*. Our interview is one way to offer such inputs. More generally, a combination of enough verbal knowledge about counting, addition knowledge, and a data structure designed to encourage thought about generating larger and larger numbers should foster induction of the natural number successor principle.

Of course children in our study have much to learn once they understand the successor

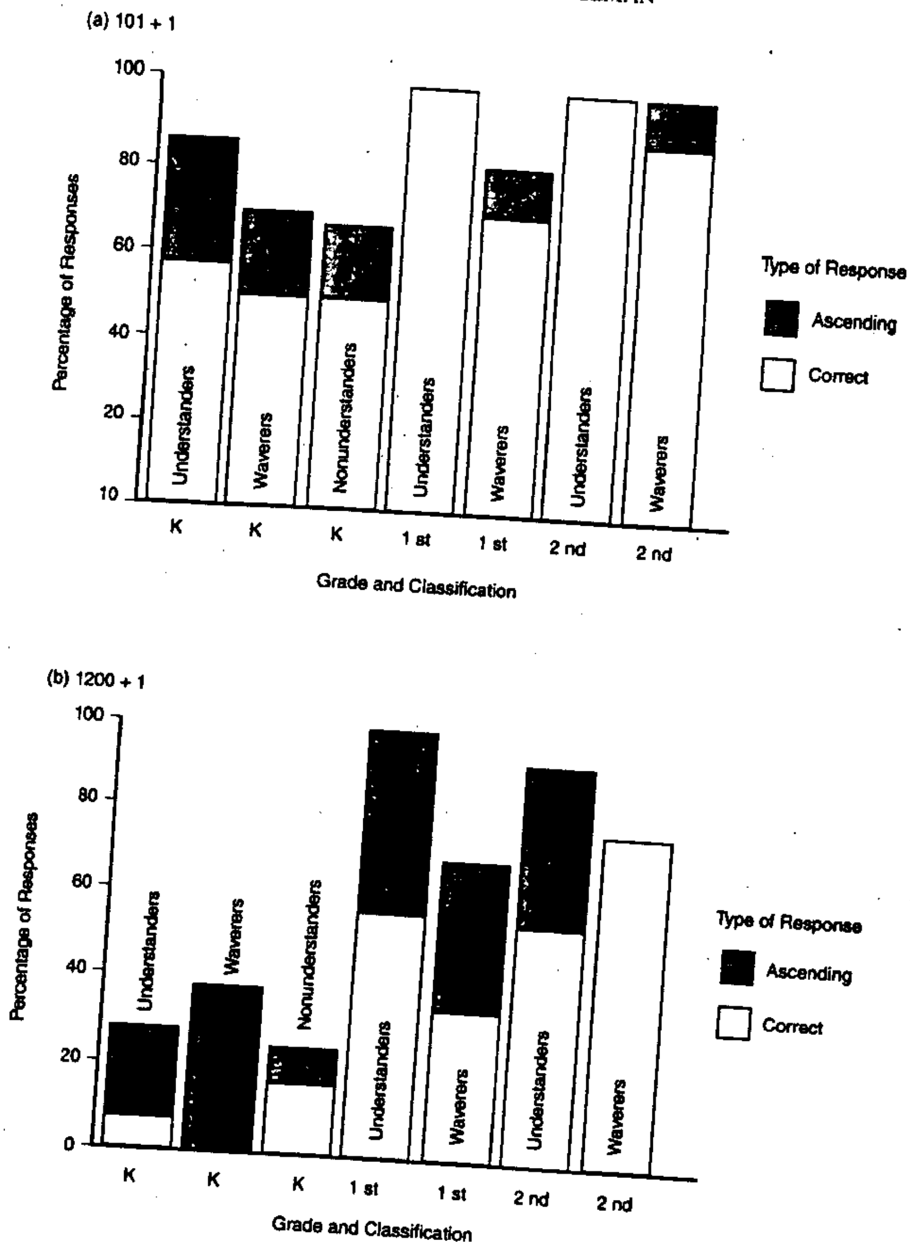

Figure 7. Percentage of subjects who gave correct or ascending sums in response to each of two addition problems.

principle. Although the children understand the processes that underlie the generation of successive natural numbers, they have a ways to go before they will be able to talk about these in the language of mathematics. When Understanders justified their Y/N answers, few were *Mathematical Infinite* explanations, where the statement that there is no last number focuses on the fact that addition is always possible, for example, "Because if you keep adding one to a number, you can keep getting a higher number. And you can keep adding one forever; you'll never get to the end" (see Hartnett, 1991 for more on this theme, especially as regard notions about infinity). Without diminishing the importance of the kind of learning children have ahead of them, one still has to be impressed with the level of accomplishments we have encountered. We spent very little time with the children on a concept that they do not receive instruction about. Their successes are even more compelling when put up against their difficulties with fractions.

The Fraction Studies

Although primary grade children seldom are offered inputs for learning about the successor principle, they are introduced to fractional representations. Number lines are an ubiquitous prop of Kindergarten classes. Blackboards prominently display fraction numergraphs and numerlogs e.g., $\frac{1}{2}$ and "one half". Textbooks include illustrated lessons on the labels of N equal parts of shapes, where N is at least 2, 3 and 4 and the related fractions, $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$. They also offer some instruction on how to talk and write about the equal parts and symbols, e.g., "this circle has 2 equal parts, each is one half of a circle. We can write *one half* as $\frac{1}{2}$", and so on. Finally, children have some opportunities to mix fractions with whole numbers, in either fractional notation or natural language (e.g., $1\frac{1}{2}$ and "one and one half").

In the previous section we showed that even though primary grade children do not get lessons on infinity and related concepts, they were quick to assimilate relevant evidence. We think this happened because the organizing principles of the relevant inputs overlap with those of the children's knowledge structures. What was known could serve as a foundation for further learning about numbers. However, if what is known is different in form and even inconsistent with the structure of what is to-be-learned, the learner could be at risk. Our research on fractions shows that this is more than a possibility.

The inputs designed to map the mathematical fact that fractions are numbers and are not consistent with the principles of counting. One cannot count things to get a fraction. The mathematical principles that gives a fraction its numberhood status are different from those that give a natural number its status as a number. Similarly, the different principles have different mapping rules for the symbol sets that are acceptable representations of the two kinds of numbers. The nonverbal counting principles do not map to the tripartite symbols that are used to represent fractions.

These considerations led us to propose that inputs about fractions may not be interpreted as intended by the school, but rather in terms of the child's theory that number is "what one gets when one counts things." If so, $\frac{1}{4}$ could be judged as larger than $\frac{1}{2}$ because "4 is more than 2". Doing this amounts to treating the data as if it were made up of novel counting examples as opposed to exemplars of a new kind of number. The consequence is that the child will not start down the learning path that leads to a mathematical understanding of fractions. We already have some evidence that the risk is real.

Gelman, Cohen and Hartnett (1989) asked Kindergarten, Grade 1 and 2 children to place circle representations of fractions on a special number line, one that represented whole numbers with sets of N circles. The majority of the children used whole number count strategies to solve the task. For example, a test display that showed $1\frac{1}{2}$ circles was placed at the 2 position. Similarly, many children placed all of the representations of unit fractions (one fourth, one third and one half of a circle) at the position for 1 on the number line, often telling us that each was "one thing". When these same children were shown two cards, one each with $\frac{1}{2}$ and $\frac{1}{4}$, they could point correctly to the card that "showed one half." Still, they systematically misordered the two fraction numerlogs, as if they were treating the displays as novel examples of counting numbers. For they chose $\frac{1}{4}$ as the larger of the two numbers. Similarly, they seldom read $\frac{1}{2}$ correctly, as "one half." Instead they came up with a variety of alternatives, including "one and two", "one and a half", "one plus two", "twelve", "three." The most frequent misreading was "one and a half", a fact that we return to below.

We were concerned that the Gelman *et al.* (1989) study unwittingly misled the children. Perhaps by not labeling our novel materials with the terms offered in their classrooms, we encouraged them to use whole circles (e.g., Siegal, 1991). Therefore, our next fraction study (Gelman, 1991) had three Training conditions. The Control Condition was run much like the first study, during all phases including the pretest, testing, and transfer phases. The two experimental groups were offered pretest experiences that used mathematical talk to introduce the children to the equal sized fractional representations of circles (and other shapes). Shapes were taken apart into different number of equal sized units and put back together to make one. Children also practiced labeling and ordering different sized unit fractions and we talked to them about the mathematical reason for a given ordering. To set up the placement task, we first introduced a number line with numergraphs (including ones that represented quantities others than whole numbers). We then covered these representations with sets of wholes and fractional parts of squares to show that the representations on the number line could be different. Finally, we introduced our special number line where sets of circles represented cardinal values. The subsequent testing conditions varied depending on whether a child was in the Label or Non-label condition. In the Label condition, children were told the numerical value of each test item at the time they were asked to place it where it belonged on the number line, e.g., "Put the one third where it belongs." Children in the Non-label condition were simply handed the cards, one at time, and asked to "Put it where it belongs". The results of this study complemented those from our first fraction study. Kindergarten and Grade 1 children had potent tendencies to use one or another kind of counting strategy. For example, when Label condition children placed a "three thirds" test item, they typically put it at the point that represented 3 "because it is three". That is, our labeling the stimulus as *three* thirds called attention to the three parts as opposed to the mathematical idea that $\frac{3}{3} = 1$. The word *three* was selectively assimilated to the child's understanding of verbal instantiations of the counting principles. Non-label children often placed the three-wedge display correctly; however, this did not reflect better mathematical understanding. Success on such items was due to their tendency to act as if they were in a perceptual closure task, that is to say that the pieces could be pushed together to make one circle. Since this is a non-numerical strategy, it is inappropriate to say such performances were correct. This observation reinforced our decision to do a study in which children were asked to place whole number and fraction numergraphs on an ordering cloth.

The Gelman, Meck and Sundaralingam Numerlog Ordering Study

Without an understanding that a fraction is the number one gets when a number is divided by another and that such numbers may be ordered, added, subtracted, multiplied and divided right along with the numbers that one gets by counting, children should have difficulty ordering fraction numergraphs. We tested this hypothesis in this study, taking care to include test items that varied in the degree to which they were likely to be familiar to the children. Therefore the placement items included examples of: (a) the unit fractions, the fractional numergraphs that are taught first, e.g., $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$; (b) representations that children actually learn first, e.g., $1\frac{1}{2}, 2\frac{1}{2}$ ... (Gelman, 1991); (c) school materials introduced after unit fractions, (e.g., $\frac{2}{3}, \frac{2}{4}, 2\frac{1}{4}$); (d) what we call mathematical wholes (e.g., $\frac{2}{2}$ and $\frac{3}{3}$); and (e) the whole (counting) numbers 1, 2 and 3.

Details of the Study

Subjects. Since the youngest children in the Successor studies had limited knowledge of the counting list, we thought it best to focus our subject recruitment on children who were at least nearing the end of the Grade 1. Equal numbers of children were recruited from two sources: a summer day camp and a private school, that offered an open classroom, progressive curriculum. The camp children were from public elementary schools and told us they had learned about fractions at school. The children at the private school had received little, if any, formal instruction about fractions. Signed permission to interview the children was obtained from 90% of our permission requests with parents at the camp. The school children's families received a mailing that included a stamped, addressed permission form. We honored the school's request to work with those children for whom a single inquiry led to a reply; the return rate was 35% (all positive), a comparable rate for the first round of other mailings we have done.

Thirty-four children, about twice as many girls as boys, who ranged in age from (years:months) 6:1 to 9:11, participated in the study. When divided into younger and older groups, their average ages and ranges were 7:6 (6:1–8:7) and 9:2 (8:8–9:11), respectively. All lived in West Los Angeles county and were from middle class backgrounds. Almost all were white; a few were black or Latino.

Procedures

The study consisted of two sessions, a pretest and a placement session, over 2 days. For those children who had the time to continue, some questions designed to get another index of their ability to order and operate with fractions were asked at the end of Day 2.

The Tasks

Stick-ordering. This pretest task was designed to encourage children to order from the smallest to largest value. Stimuli consisted of a pile of orange sticks of the following lengths in inches: 1, 3, 4, 6, 7, 9, 10, as well as a duplicate of two of the middle length sticks. To start the stick-ordering pretest task, the experimenter selected the longest and shortest of these and put them down on the ordering cloth (about 75 × 275 cm) and said "I brought some sticks and we are going to play an ordering game with them. This is the longest stick and it goes here" (in front of the child) and this is the shortest stick and it goes here (placed to the left of the first stick and spaced so that only one stick could fit in between the two sticks). "I am going to give you some more sticks and I want you to put them where you think they belong. Remember that the sticks go in order with the longest here and the shortest here". If the child erred, the experimenter was allowed to say "the longest stick is here and the shortest is here and you have to put them in order; so where does this stick go?" Similarly, since we wanted the children to understand that they were to order from smallest to largest, experimenters had examples of hints they could use. At first the child was asked to explain their placement; this can lead to the recognition of and correction of an error. A second approach involved asking the child to judge which of a pair of sticks was longer or shorter, reminding the child that she was to order the sticks from shortest to longest, and suggesting she move the incorrectly placed stick. If these hints were not successful, the experimenter finally showed the child the correct place for the stick in question (of course, this placement was not counted as correct).

After children correctly placed a sample of each of the different length sticks, they were given a duplicate of a middle size stick and asked to put it where it belonged. This was done to encourage the idea that variants of equal value could be placed in the same rank order position, either above or below already-placed items. The experimenter asked children to explain these placements so as introduce a chance for her to say that equally long sticks could go at the same place: "When they are equal, it (the second stick) can go here".

The length ordering pretest task was run a second time, with one change. A different stick serve as the duplicate. The session lasted about 10–15 min. During the first trial, all but two children placed sticks correctly; no one erred on the second trial. When explaining their placements, all children mentioned a stick's length in relationship to other sticks already on the cloth.

Numergraph placement. To start the second day of testing, the experimenter said "Now we are going to play another ordering game with the same rules but with different things to put in order. You may not have seen these things before but that's OK. When we play this game with other children your age, they had fun". Children then were asked to identify 1, 1 ½ and 2. All could order these three symbols without help. Similarly, all children could order and justify correctly their placements on the ordering cloth of the numerlogs for the natural numbers 0 through 4.

To start the experimental phase proper, the experimenter removed the numergraphs from the pretest ordering cloth. She next showed all of the test items in a face down position, so as to give children a sense of how many cards were in the placement set. They were told that they would get a chance to place each one "where it went best." The procedure proper began when the experimenter placed 0 to the far left of the cloth and asked the child to place the number 4 on the cloth "where it belonged". The test cards then followed, one at a time, in a different random order for each child. These had the following numerlogs on each of them:

$$\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{4}, \frac{3}{4}, \frac{2}{2}, \frac{3}{3}, 1, 1\frac{1}{4}, 1\frac{1}{2}, 1\frac{3}{4}, 2, 2\frac{1}{3}, 2\frac{1}{2}, 2\frac{2}{3}, 3, 3\frac{1}{2}$$

As the children were shown each card, they were asked to read it, place it where it belonged on the cloth, and then justify their placements. If a child had difficulty reading the cards, the experimenter helped. For example, for the card containing $1\frac{1}{4}$, the experimenter would cover the $\frac{1}{4}$ and say "this is one." She would than remove her hand and say "this is $\frac{1}{4}$ and if you put it together the card says 1 and $\frac{1}{4}$".

Analyses and Results

Numergraph reading skill. Only 2 of the 34 children read all of the fraction numergraph cards correctly. Another 18 read at least 79% of the cards correctly without assistance. The overall percentage correct reading score for each fraction is shown in Fig. 8.

The children were best at reading cards with numerlogs representing values halfway between whole numbers; the stimuli $1\frac{1}{2}$ and $2\frac{1}{2}$ yielded the two highest correct reading

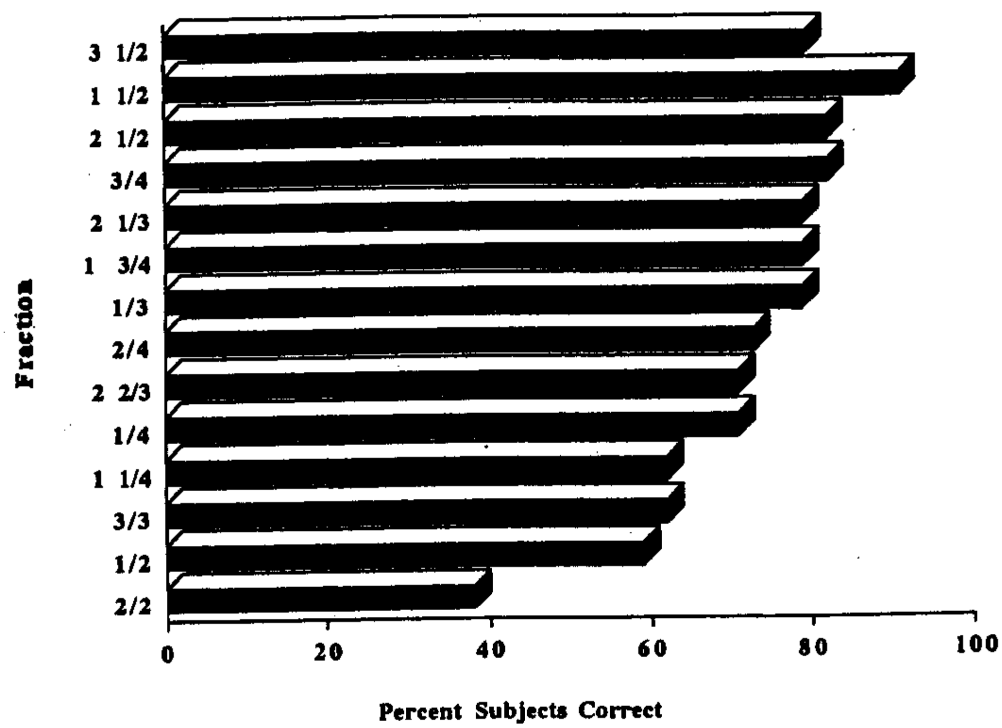

Figure 8. Percentage of children who read each of the test numergraphs correctly in the fraction study.

scores, 91 and 82%, respectively. This fits with Gelman's (1991) training effects; during transfer, the children were best able to place novel items that represented the number halfway between representations of two cardinal numbers. As in that study, the children in the present study often made reference to the fact that one can count in a way that is consistent with this kind of behavior, for example "One and half comes after one; a half of something doesn't say [anything], just a half — should say zero and a half. It [a half] could go anywhere". So, caution has to be excerised when we interpret as "correct" children's placements of $1\frac{1}{2}$ and $2\frac{1}{2}$, etc. It is possible that these placements reflect a willingness to extend the count list, as opposed to an understanding of fractional representations. To be sure, such a strategy is an assimilation that is commensurate with the idea that fractions are numbers. But it is still necessary for children to learn to understand that they are fractions and not count numbers.

The placement numergraphs that represented non-unit fractions — $\frac{3}{4}$ and $\frac{3}{4}$ — were read about as well as the mixed numergraph representations for whole numbers and non-unit fractions — $1\frac{1}{4}$, $1\frac{3}{4}$, $2\frac{1}{3}$ and $2\frac{2}{3}$. The respective correct readings were 74 and 73%. Children did not do especially well with $\frac{2}{2}$ and $\frac{3}{3}$ (mathematical wholes). The percentage correct for all children was 50 for each of these. When we take age into account, the younger children had more difficulty with $\frac{2}{2}$, presumably because of limited experience with such entities.

Finally, and unexpectedly, the numergraph $\frac{1}{2}$ gave the children a considerable amount of difficulty. Only 41% of the younger group read it correctly and even some (24%) of the older children erred. The most frequent misreading error for $\frac{1}{2}$ was "one and a half". A similar result is reported in Gelman (1991) where we introduced the idea that children first think of "a half" as something that is relevant to expanding the counting list and not as a representation for a number between 0 and 1. For them the numbers have to start at one (it is not possible to count to get zero). The problem then is to assimilate things like "one half" or $\frac{1}{2}$ to this notion of what a number is. Reading and placing $\frac{1}{2}$ as if is between 1 and 2 is one way to accomplish this. We suggest that the fact that $\frac{1}{2}$ contains the whole numbers 1 and 2 reinforces this move and helps account for its ubiquity. The pattern of children's placements adds weight to our idea that they were constructing ways to treat representations of non-counting numbers as novel examples of counting numbers.

Placement skill. No child was able to place all items in a correct order. For reasons just considered, our first analysis of children's patterns of answers considered how well subjects would have done if they only had to place 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3 and $3\frac{1}{2}$. To answer this question we asked what order these items would occupy if we removed all other items in a child's solution. Therefore, a pass for this kind of analysis meant that a child had rank ordered the pairs (1 and $1\frac{1}{2}$), (2 and $2\frac{1}{2}$) and (3 and $3\frac{1}{2}$) and the items within each pair. Fifty-three and 76% of younger and older groups, respectively, met this criterion; the age difference is not reliable ($\chi^2 = 2.06$, $df = 2$, $p > 0.05$). This is one indication that children favored the use of a count-based strategies. Their comments reinforce the interpretation. For example, one child placed $\frac{2}{3}$ before $\frac{3}{4}$ and said "2 is less than 3 and 3 is less than 4". Another child said "$\frac{1}{3}$ before $\frac{1}{4}$", presumably because the denominator of the latter represents a larger cardinal value. Yet another child who placed $\frac{3}{3}$ after 3 helps us illustrates the same point: "That [$\frac{3}{3}$] is bigger than 3; you have made 3 and are trying to make 4 now". She first compared the cardinal value of the numerator with the whole number 3 and then cardinal value of the denominator with the whole number 4.

However charming the explanation, she clearly treated the fractional numergraph as another kind of counting numerlog as opposed to a fraction that stands for a mathematical whole.

Placement response rules. We reliably classified the pattern of children's placements by using a combination of two rules: two raters agreed for 95% of the cases. Rule I states as to whether children kept whole numbers separate from fractions or interposed the numergraphs for both kinds of numbers. They succeeded in separating them in one of two ways: (a) by placing all, or most of the whole number numergraphs in order and to the left. All other stimuli were placed to the right of this group; or (b) by placing nothing between 0 and 1. This meant that *all* of the test items, be they numergraphic representations of whole numbers, unit fractions, mathematical wholes or mixed numbers, were placed to the right of 1. Rule II summarizes children's tendencies to generalize their ordering knowledge of natural numbers to the task of ordering fractional numergraphs. The natural number ordering rule had three variants. These are illustrated in Table 7 and were:

1. *Edge-matching*, which is an immature classification strategy identified by Bruner, Olver, Greenfield *et al.* (1966) who observed young children start to pair items that shared a common attribute and then continue to switch their focus of attention to some new attribute of the second or third item and therefore add to their "classification", a new object that matched the newly attended-to attribute. An example of such a "classification" is a line of blocks wherein the first two are red squares, the third is a red circle, the fourth a yellow circle, the fifth a yellow triangle, and so on. In our placement task, children who edge-matched placed successive pairs or triplets that shared a common natural number alongside each other. Within pairs or triplets in turn were rank ordered with respect to a locally shared number.
2. *Similarity clustering*, where children placed a large number of stimuli next to each

Table 7
Examples of Placements that Illustrate the Rules used to Classify Individual Children's Overall Patterns

Pattern	Examples
Edge-matching	Example 1: nothing between 0 and 1
	$0\ 1\ 2\ \frac{1}{2}\ 1\frac{1}{2}\ 2\frac{1}{2}\ 2\frac{2}{3}\ \frac{1}{3}\ \frac{3}{3}\ 1\frac{1}{4}\ \frac{1}{4}\ \frac{2}{4}\ \frac{2}{2}\ 1\frac{3}{4}\ 2\frac{1}{3}\ 3\frac{1}{2}\ \frac{3}{4}\ 3\ 4$
	Example 2: whole numbers first
	$0\ 1\ 2\ 3\ 4\ \frac{1}{2}\ 2\frac{1}{2}\ 2\frac{1}{3}\ \frac{3}{3}\ \frac{2}{2}\ 1\frac{1}{2}\ 1\frac{1}{4}\ \frac{2}{4}\ \frac{3}{4}\ 3\frac{1}{2}\ 2\frac{2}{3}\ \frac{1}{3}\ 1\frac{3}{4}\ \frac{1}{4}$
Cluster	Example 1: nothing between 0 and 1
	$0\ 1\ \frac{1}{2}\ \frac{1}{3}\ \frac{1}{4}\ 1\frac{1}{2}\ 1\frac{3}{4}\ 2\ \frac{2}{2}\ \frac{2}{4}\ 2\frac{1}{2}\ 2\frac{1}{3}\ 2\frac{2}{3}\ 3\ \frac{3}{3}\ \frac{3}{4}\ 3\frac{1}{2}\ 4$
Whole number ordering	Example 1
	$0\ \frac{1}{2}\ \frac{2}{2}\ \frac{1}{3}\ \frac{3}{3}\ \frac{1}{4}\ \frac{2}{4}\ \frac{3}{4}\ 1\ 1\frac{1}{2}\ 1\frac{1}{4}\ 1\frac{3}{4}\ 2\ 2\frac{1}{2}\ 2\frac{1}{3}\ 3\frac{1}{2}\ 4$
	Example 2: nothing between 0 and 1
	$0\ 1\ \frac{1}{2}\ \frac{1}{3}\ \frac{1}{4}\ 1\frac{1}{2}\ 1\frac{1}{4}\ 1\frac{3}{4}\ 2\ \frac{2}{2}\ \frac{2}{4}\ 2\frac{1}{2}\ 2\frac{1}{3}\ 2\frac{2}{3}\ 3\ \frac{3}{3}\ \frac{3}{4}\ 3\frac{1}{2}\ 4$

other that looked like alike in terms of their degree of "oneness", "twoness", and so on. See Kemler (1983) for a discussion of overall similarity organizations;

3. *Systematic whole number counting*, the use of which involved the rank ordering of fractions on the basis of the cardinal value of the denominator if possible. If there was no clear way to use the denominator, children switched to rank ordering on the basis of the numerator. For example, children would place $\frac{1}{2}$ to the left of $\frac{1}{3}$ and tell us that 3 is bigger than 2, or place $\frac{2}{2}$ before $\frac{3}{4}$, now because "2 and 2 are less than 3 and 4".

Fifty-three percent of the children produced solutions that started the rank ordering either with 0 and 1 or all of the whole numbers, as if either 0 and 1 or 0, 1, 2, 3 and 4 were in categories that differed from ones with any that had fractional representations. Only one child who used a systematic counting rule did this. Children's tendency to separate whole numbers or avoid putting fractions between 0 and 1 was reliably related to whether they used an edge-matching, clustering or systematic counting strategy ($\chi^2 = 5.42$, $p < 0.02$). Findings like these fit our proposal that, when children start to accept the idea that there might be numbers between 0 and 1, they first try to make sense of them in terms of their current theory of number. Systematic count solutions are used to accommodate the need to place representations of numbers between 0 and 1 and whole numbers. As creative as these adaptations are, they nevertheless are wrong and will have to be abandoned in favor of different ideas about fractions and their written representations.

Summary

If the structure of inputs does not map easily onto existing structures, then learning can be at risk. Such a mismatch occurs when children are supposed to learn about fractions; the counting principles are not consistent with the mathematical principles that apply to the definition of a fraction.

Conclusions and Conjectures

We have illustrated the two-edged consequence of the constructivist mind by comparing children's responses to inputs about infinity and fractions. Because counting principles are consistent with the Successor principle, learning proceeds rather smoothly. In contrast, since the mathematical notion of a fraction is inconsistent with the idea that numbers are what one gets by applying the counting principles, we should not be surprised if fractional representations are misinterpreted. Children's counting principles do not suffice for the accurate uptake of the data. There is no isomorphism between the systems of counting and adding that they know and the relevant characteristics of the fraction data. To conclude, we turn to the question of how children might ever get beyond these mental barriers and acquire new mathematical understandings.

In one sense, the solution is easy. Teach children new conceptual structures. This suggestion is hard to put into practice, for it requires us to offer relevant input and create the structure that will support understanding at the same time. We suggest that one way to do this is to pair the instruction of the language of fractions with the conceptual system that makes the language and its references meaningful. Historians of science repeatedly note how the meaning of mathematical and scientific terms is related to the concepts and the principles that organize these terms in a given domain of science (e.g., Kitcher, 1982; Kuhn, 1970). Advances in our understanding of scientific concepts, therefore, move in concert with our acquisition of the language that refers to these. Learning about the language of science and learning about the concepts of science can be thought of as two sides of one coin: progress in learning about one bootstraps progress in learning about the other. Inputs need to be organized to vividly represent and thereby cue in memory the relevant facts and principles as well as the appropriate language and word meanings. Although it is hard to accomplish this task, it is not impossible.

Another way to frame the young child's learning problem is to note its resemblance to developments in the history of science and mathematics. The meanings of the term "number" in the system of natural numbers and in the system of rational numbers are incommensurate. Fortunately, however, the meanings of the count words themselves are not incommensurate in the two systems; the integers have the same properties in the system of rational numbers that they have in the system of natural numbers. If one takes advantage of these local commensurabilities, one gets a foot in the door that opens to the new concept. Remarkably, some children do just this. They find places where what they know and what they have to learn in school are at least locally commensurate and try to build from there. Here and in our previous work, we find that, with experience, some children come to talk of numbers between "one" and "two", even if they do not know the mathematical meaning of the terms they use. Recall how many of our subjects ordered 1, 1 $\frac{1}{2}$, 2, 2 $\frac{1}{2}$, 3 and 3 $\frac{1}{2}$ correctly. Similarly, some children accepted the idea that they could place fractions representing values less than 1 between 0 and 1. These solutions are at least consistent with the induction that more "numbers" could come between each of those in the count list. As children continue to add names and representations of these kinds, they also create databases that can support relevant inductions, for example that "numbers" could come between those in their extended count list, and even between 0 and 1. These ideas, as incomplete as they might be, are consistent with the idea that there can be many, many, many, in fact, indefinitely many numbers between a given pair of numbers. Therefore, although children's acceptance of the idea that there are numbers between 0 and 1 and each of the counting numbers is ripe with problems, it still can serve instructional goals. It becomes possible to introduce the idea that there is no referent for "the next number after one and a half", that the Successor principles do not hold when one extends the number system to include fractions, and that fractions are the kind of number one gets by dividing one number by another.

For us these ideas amount to a defense of instruction that pairs use of the abstract with the concrete, talk about the meaning of mathematical terms at the same time that students work with materials that can help them learn about the new mathematical concepts. Pairing the abstract with the concrete is a salient characteristic of those programs that are heralded for their success at teaching mathematics. Differences between the way 6- and 7-year-olds in Japan and the United States are taught mathematics illustrates this (Stevenson & Stigler, 1992; Stigler & Fernandez (1995). In the United States, children might be introduced to these facts with any of a series of supporting materials. The idea is to introduce symbolic ideas with concrete, physical examples. Comparable materials will be found in a Japanese classroom but they will be used in a different way, as part of mathematical discussions about possible solutions. Coherence in the lesson is derived from this discussion and from the successful integration of the concrete and abstract data about the relevant principles of mathematics. Answers serve as occasions to encourage children to think of different ways to state the mathematical problem, for example that:

$$(3 + 4) = 1 + 1 + 1 + 1 + 1 + 1 + 1 = (2 + 2) + 3 = 5 + 2...$$

Nesher and Sukenik (1991) have developed the idea that students in math classes benefit from abstract inputs that organize examples in their effort to deal with the difficulty high school students have with formal representations of ratios and proportions. Their goal was to offer instruction to support acquisition of both implicit and explicit understanding. The latter is a necessary foundation if students are to learn more mathematics. The students started with explicit representations of all entities in the expression $a/b = c$. This was accomplished in a color mixing demonstration. Quantities of blue and yellow were mixed in water to generate green. Variations in the amounts of yellow yielded lighter and darker shades of green. Students predicted how dark the green water would be as a result of mixing different quantities of yellow and blue. Because they received immediate feedback, students had an opportunity to compare their predictions with the outcome and gain relevant information about ratios. This feedback enhanced learning to some extent, but the learning was most enhanced when these experiences were paired with the parallel relevant formal statements. The formal representations of rations and proportions offered a particularly apt structure within which to summarize the findings of the experiment in a meaningful way.

Finally, we note that young children can be active participants in mathematical conversations, especially when they have implicit understanding of the concepts involved. Our young subjects became mathematical conversationalists when given the opportunity to talk about the Successor principle. Perhaps we underestimate these young learners when we cling to the idea that they are not ready to learn about the abstract. What matters more is whether inputs, no matter at what level of concreteness or abstractness, can be mapped to existing knowledge structures. Abstractness in and of itself need not be a problem for young learners. The barriers to learning are better understood as mismatches between the structure of what is known and what is to be learned.

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