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Foreword (pages xii-xix)

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It is a pleasure to introduce what is sure to become a prominent mark of progress in the scientific understanding of the origins of numerical cognition. This volume provides a comprehensive view of that progress, revealing the emergence of a broad consensus on the answers to several foundational questions, a consensus far removed from the views commonly entertained half a century ago. A half century or so ago, it was commonly assumed that numerical cognition depended on language for its development and that it developed, therefore, only in humans. We now understand that numerical cognition is evolutionarily ancient. Many of the chapters in this volume review aspects of the empirical basis for this new understanding, but the reader is directed most particularly to the first four chapters and to the final chapter. We also now understand that the pre-linguistic system of numerical estimation and reasoning, which we share with most if not all vertebrates (and possibly with invertebrates), plays an important role in the development of verbalized mathematical reasoning. A majority of the chapters touch on the evidence for this latter conclusion, but the reader is referred most particularly to Chapters 5 through 10.

In this introduction, we begin with a brief history of contrasting ideas about what a number is. We do so to clarify for the reader some important distinctions, most importantly the distinction between numbers as symbols for discrete (countable) quantities and numbers as the players in the system of arithmetic reasoning. Gelman (1972) referred to this as the estimator-operator distinction.

Throughout most of the history of mathematics, numbers were defined by what they referred to, which is to say countable quantities. The traditional view is memorably summed up by a well known quote attributed to Kronecker, "God made the integers; all else is the work of man." (quoted in an obituary by Weber 1893)

Only with the extensive formalization of mathematics in the latter part of the 19th century did an alternative view gain currency, the view that now dominates among mathematicians. This view is articulated by Knopp (1952, p. 5), who gives an axiomatization of arithmetic followed by: "Once the validity of these fundamental laws has been established, it is unnecessary, in all further work with the literal quantities a, b,…, to make use again of the fact that these symbols denote rational numbers. … From the important fact that the *meaning* [that is, the reference] of the literal symbols need not be considered at all…, there results immediately the following extraordinarily significant consequence: If one has any other entities whatsoever besides the rational numbers, …, *but which obey the same fundamental laws*, one can operate with them as with the rational numbers, according to exactly the same rules. Every system of objects for which this is true is called a *number system*, because, in a few words, it is customary to call all those objects *numbers* with which one can operate according to the fundamental laws we have listed." (italics in original).

On this *formalist* view, numbers are defined in the same way that chess pieces are, that is, by what can be done with them. The empirical reasons for regarding the brain's non-verbal representation of discrete (countable) quantity as part of a more general system of arithmetic reasoning that represents *both* continuous *and* discrete quantity is the focus of Chapter 6 (see also Walsh 2003; Gallistel 2011). The fact that a system of symbols as simple as arithmetic can so effectively represent so much of experienced reality is astonishing (Wigner 1960). The representational power of arithmetic reasoning probably explains why it emerged so early in evolution.

Our intuitive identification of number with *countable* quantity is evident already in the troublesome ambiguity about what exactly an author may mean when using 'number' in many contexts. Do they mean to refer to the numerosity of a set? Or do they mean to refer to a conventional symbol, such as 'two' or '11' or 'IV' or (5-2), that may (or may not) represent the numerosity of a set? The distinction here corresponds to the distinction in computer programming between, on the one hand, the bit-code representation of a number that enters into the arithmetic operations that a computer performs and, on the other hand, the bit-code representation of the textual symbol for that number. This distinction (between x = 7 and x = '7') often flummoxes newcomers to the programming craft. It is now common in the technical literature to use 'numerosity' when the first meaning is intended. Gelman and Gallistel (1978) suggest 'numerlog' when reference to the conventional symbols is instead intended, but that coinage has not come into common use. In any event, failing to distinguish between the numerosities themselves, on the one hand, and the symbols for quantities in an arithmetic system, on the other, may lead some readers into confusion.

Some of the current authors use 'symbolic' to refer to conventional symbolizations of quantity, particularly when asserting that the evolutionarily ancient system is "non-symbolic" or that only humans have a "symbolic" representation of number. We would suggest that this usage invites confusion. It seems to imply that the evolutionarily ancient system does not employ symbols in the sense in which a computer scientist understands this term (see Gallistel and King 2010). Insofar as someone subscribes to a radical connectionist view of the brain, a view in which brain activity is "sub-symbolic" and does not in any true sense represent the experienced world, this might be the intended meaning, but we do not believe it is the meaning intended by any current author. To distinguish between conventional symbols and the brain's symbols, we suggested 'numerons' to refer to the brain's symbols for quantity (Gelman and Gallistel 1978), but again that coinage has not gained currency.

Symbols are essential elements of representations (Gallistel and King 2010). A representation has two fundamental components, a component (set of processes) that map from aspects of the represented system (for example, from numerosities) to the symbols that thereby refer to them and a component (set of processes) that operates on the symbols (ordering them, adding or subtracting them, multiplying or dividing them, concatenating or extracting them, and so on). Symbols are the stuff of computation, and representations are constructed by computation. That is the essence of the computational theory of mind, which is the central doctrine of cognitive science. If the brain represents quantities—and, in our view, the chapters in this volume show beyond reasonable argument that it does—then it does so by means of symbols.

Because the meanings of numerical symbols were once commonly thought to derive from their reference, there were doubts about whether negative numbers and irrational numbers really existed, even among professional mathematicians living into the 19^{th} century (Kline 1972). However, by then, the use of numerical coordinates to represent different aspects of the physical and financial world was rapidly advancing. Because the origin of a useful coordinate system is almost always arbitrary (the Greenwich meridian, for example) and/or because many of the quantities represented by numbers are inherently directed quantity (distance north and south, for example), it is all but impossible to avoid using negative numbers in practice. Moreover, in algebraic reasoning, one often does not know until it is, so to speak, "too late," whether one is or is not trafficking with negative quantities. Thus, some of what Kronecker regarded as the works of man (the negative numbers and fractions) seemed unavoidable if one were to have a generally useful system for representing quantities, as even Kronekcer conceded. Kronecker agreed that negative numbers and fractions could be admitted into the system made by God (the counting numbers), provided that it could be proven that they could be made to abide by the manipulation rules that God's creatures (the so-called natural numbers) abided by. However, the enlargement of the system of what people took to be the "natural" numbers to make it more generally useful led most mathematicians to include among the numbers the so-called transcendental numbers like π, whose very existence Kronecker disputed. (He insisted that there was no such number!) One begins to see why the question whether a symbol did or did not abide by the rules of arithmetic came to be an ever more important consideration, emerging eventually as definitive of number itself.

Theories of the development of verbalized numerical reasoning divide along similar lines. Some authorities urge that the preschool child understands the meaning of 'one' if and only if s/he knows that this word refers to the numerosity of sets that have only one member, and similarly for the meaning of 'two' and 'three' (Carey 2010). This view is similar to (and perhaps partially motivated by) Gottlob Frege's attempt to found mathematics on logic and set theory by defining '0' to *mean* (that is, denote or refer to) the numerosity of the empty set, '1' to *mean* the numerosity of sets that could be placed in 1:1 correspondence with the set that contains only the empty set, '2' to *mean* the numerosity of sets that can be placed in 1:1 correspondence with the set that contains only the empty set and the set that contains the empty set, and so on (Frege 1884). On this first view of the development of verbal numerical reasoning, sets composed of the pointer symbols in the object tracking system (OTS) serve in place of Frege's recursively defined canonical sets. Reference to canonical pointer-sets in long term memory is what gives the first few number words their meaning in the mind of the very young child, on this view.

Other authorities urge that the non-verbal system of numerical reasoning (generally called the ANS, which is short for analog number system) is the principal foundation for the development of the human capacity to reason verbally about quantity (Gallistel and Gelman 1992 and Chapters 5-10 of the current volume). This latter view stresses that the understanding of number words always involves an appreciation of two facts: (1)[ordinary] number words refer to countable quantities; (2) they also correspond to preverbal symbols in a system of preverbal arithmetic reasoning. On the first view, the child in the early stages of learning to count does not grasp that it is learning words for mental symbols that are subject to arithmetic reasoning, symbols that can be ordered, added and subtracted. On the second view, the child in the early stages of learning to count already understands that number words refer to symbols that are subject to arithmetic reasoning. In other words, the child just learning to count already understands that number words also correspond to players in the game of arithmetic, a game its brain already knows how to play. Several chapters in the current volume review the now extensive evidence that the brains of infants and toddlers already play the number game when thinking about both countable and continuous quantities (Chapters 5-10). Other chapters review the evidence that the brain's ability to play this game has ancient evolutionary origins (Chapters 1-4 and Chapter 13, see also Livingstone 2014).

The question of the extent to which the development of numerical reasoning depends on the realization that number words correspond to entities that are subject to arithmetic reasoning bears crucially on proposals that there is more than one non-verbal system for representing numerosity. A profoundly important formal property of the number system is *closure*. A symbolic system is closed if the valid operations on its symbols always yield other symbols within that same system. When it is proposed that there is one non-verbal system for representing small numerosities (usually 4 or less) and a different non-verbal system for representing larger numerosities, one must ask why this does not lead to lack of closure in both systems. The problem is that 2 + 3, which is an operation on symbols for small numerosities, yields 5, which is not a symbol in the putative system for representing small numbers. Similarly, 7–5, which is an operation on symbols for large numerosities, yields 2, which, at least on some stories, is a symbol that exists only in a different system. It is as if a move in chess carried you out of the game of chess and into the game of checkers. But even a parrot can do 2+3 = 5 (see Chapter 3).

The closure consideration does not argue that, for example, the object tracking system (OTS for short) does not exist. There is little doubt that it does exist. And, as Chapter 10 discusses at length, it does sometimes control behavior in tasks that the experimenter intended to tap an understanding of number. The closure consideration argues only that the OTS should not be regarded as in any sense a numerical representation, because a numerical representation requires both symbols that may refer to quantities and arithmetic operations on those symbols. In any such system, closure is a major consideration. Absent closure, the system will crash whenever it processes two symbols that cannot be combined arithmetically within the limits of that system. Chapter 7 offers a novel, and plausible way of dealing with this issue, while Chapters 8-10 review the extensive evidence for non-verbal arithmetic reasoning in infants and toddlers. (See also Zur and Gelman 2004.)

As emphasized in most of the chapters in this volume and discussed at length in Chapter 12, a signature of the ANS is that it represents even discrete (countable) quantity with fixed percent resolution, that is, in accord with Weber's Law. This property is often explained on the assumption that the brain's symbols for quantities are noisy signals, with the noise scaling with signal magnitude. If the Weber Law property is really ascribable to scalar noise, then it is a bug or limitation of the system. Gallistel (2011) suggested that Weber's Law may be a feature, not a bug. It may reflect an automatic scaling mechanism such as is built into most contemporary electronic instruments to enable them to give useful measurements over many orders of magnitude. Following up on this suggestion, Wilkes (2014, under review) shows that any system for representing the computable numbers over many orders of magnitude that makes efficient use of limited physical resources must exhibit Weber's Law. Both the principle and the need for it are illustrated by scientific number notation, which uses some fixed number of digits to represent any number (hence fixed percent resolution), together with a fixed number of digits to represent the scale (order of magnitude). Livingstone et al (2014) remind us that autoscaling (aka normalization) is a well established neurobiological mechanism (see also Brenner, Bialek et al. 2000).

Two other points of broad significance run through the chapters of this volume. The first is that while number is indeed a highly abstract, fundamentally amodal aspect of our experience (see particularly Chapter 11), it is not derived from concrete sensory experience through some process of abstraction and induction, as has traditionally been supposed in empiricist philosophy and in much psychological work on number. Rather, it is a foundational aspect of the brain's capacity to represent the experienced world. The second point, one particularly stressed by Vallortigara in Chapter 2 and adopted by van Marle in Chapter 7, is that number knowledge is Exhibit A in the argument that there are core knowledge domains. These domains may be thought of as skeletal structures (Gelman 1990; Gelman 1993; Gelman and Williams 1998) that organize sensory input and direct attention in such a way as to make possible the acquisition of further knowledge, such as, for example, knowledge of how to count in one's native language.

We hope these introductory remarks will help the reader to appreciate the broad and deep importance of the rich empirical and theoretical results reported in the following chapters.

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