Counting Pairs

The *at most n*, *at least n*, and *exactly n* procedures all call for lists of identities or non-identities, at the end of the statements. For example, (1) would be symbolized as (2):

(1) There are at least three students. S^1: is a student
(2) (∃x) (S^1x & (∃y) & S^1y & (∃z) (S^1z & x≠y & x≠z & y≠z)))

According the procedure for *at least n*, we have to end the symbolization with conjunctions of the non-identities of each pair of variables. Similarly, we have to end symbolizations of *at most n* with disjunctions of the identities of each pair of variables. Keeping track of all these pairs can become tricky, especially if we have to symbolize larger numerical adjectives. If we had to symbolize, say, at least six, it would be quite hard to make sure we had remembered every single pair of variables. Fortunately, there are easy formulas that can tell us how many pairs of variables we have to account for.

At least n

There should be **n(n-1)/2** pairs of non-identities.

At most n

There should be **n(n+1)/2** pairs of identities.

Exactly n

There should be **n(n-1)/2** pairs of non-identities and **n** pairs of identities.

Combinatorics (optional)

We might wonder how these formulas are derived. In the branch of mathematics called combinatorics, one often asks the following question: given a set with n members, how many subsets are there of size k? The answer is given by the formula *n choose k*:

***n choose k* = n! / ((n-k)! k!)**

where n! denotes *n factorial* which equals n(n-1)(n-2)...(3)(2)(1).

We notice that the question of how many pairs of variables are needed is to equivalent to the question of how many 2-element subsets (i.e. pairs) are there of a set of n elements (i.e. the set of variables). This answer is given by *n choose 2*, as our subsets are of size 2. According to the formula above, *n choose 2* is

n! / ((n-2)! 2!)

This can be simplified if we notice that n! = n(n-1) ((n-2)(n-3) ... (2)(1)) = n(n-1)(n-2)!, and that 2! = (2)(1) = 2

(n(n-1)(n-2)!) / ((n-2)! 2!) = n(n-1) / 2! = n(n-1) / 2

which is our formula for the number of pairs needed in *at least n* statements.

Why then do we need n(n+1) / 2 pairs in *at most n* statements? The relevant difference between *at most n* and *at least n* statements is simply that *at most n* statements have n+1 variables, (see section 16.4.2), while *at least n* statements have n variables. Thus instead of using the formula for *n choose 2* we simply use *n+1 choose 2* instead.