Natural Deduction

Constructing truth trees is not the only method for determining whether arguments are valid; another method is known as natural deduction. To prove an argument is valid using the truth tree method, we list the premises and the negated conclusion. We then apply certain rules to the sentences until we are left with only atomic statements. If there are atomic statements that contradict each other on every branch, then we have shown that it is impossible for the premises to be true and conclusion false; i.e. we have shown that the argument is valid.

Proofs in a natural deduction system follow a different form. We list the premises, but not the conclusion. We then apply natural deduction rules to the premises until we are able to write the conclusion of the argument. The natural deduction rules are truth preserving, thus, if we are able to construct the conclusion by applying them to premises, we know that the truth of the conclusion is entailed by the truth of the premises, and so the argument is valid. If we cannot derive the conclusion from the premises however, we cannot conclude anything; the argument may indeed be invalid, but then again we may have simply failed to find a way to derive it.[1] (In this respect, truth trees are more powerful than natural deduction.)

Natural deduction has the advantage of representing a rational train of thought in that it moves linearly from the premises to the conclusion. It resembles our normal reasoning more closely than truth tables and truth trees do. For example, in evaluating your friend's argument, most likely you think about whether her conclusion follows from her argument, or whether she has a gap in her reasoning. Chances are, though, that you don't try to find a contradiction between her premises and the negation of her conclusion. Natural deduction mimics the former kind of reasoning, and is thus called *natural* deduction.

Basic Rules[2]

Reiteration (R)

α
$\therefore \alpha$

[1] The system of natural deduction presented here is sufficient to prove that any valid argument is valid; there is a systematic (though tedious) method that will lead to a derivation providing one exists. This method will not terminate until a derivation is found though, so even if we were to follow it, we could never be sure that the argument was invalid, as perhaps we had just not been working long enough to find a derivation. That systematic procedure results in unnecessarily long proofs, and so we will not use it here.

[2] The system presented here is adapted from the one presented in *The Logic Book*, Bergmann, Moor, and Nelson, 1980.

Reiteration is the simplest rule — it allows us to deduce a sentence in PL from itself, or in other words, to *reiterate* the sentence.

In the truth tree system, every connective in PL has a single rule associated with it. The rule allows us to break the connective down into its parts, i.e. to eliminate the connective, leaving only the two statements it joined. If we repeat this process enough times, we are left with atomic statements among which we look for contradictions that will close the tree. If our natural deduction system only had rules that let us eliminate connectives, though, we would never be able to derive complex statements as we would have no way of *introducing* connectives. To avoid this predicament, every connective has two types of rules associated with it: one to eliminate the connective and another to introduce it. The first type of rule works rather like the truth tree rules and is known as the Out rule. The second kind, the In rule, is less familiar.

The rules for conjunctive statements are as follows:

Simplification or Ampersand Out (&O)

   α & β
∴ α

or

   α & β
∴ β

Conjunction or Ampersand In (&I)

   α
  β
∴ α & β

These rules allow us to deduce the statements following the "∴" from those preceding it. Notice that — just as in our truth tree system — when we apply these rules the variables "α" and "β" may stand for any sort of statement – simple or complex. So, for example, we can use &I to make the following derivation:

1) S & D	premise
2) (K v H) & ~W	premise
3)∴ (S & D) & ((K v H) & ~W)	1, 2 &I

The only requirement for using these rules is the ampersand that is either eliminated or introduced must be the main connective in the sentence. This holds for the other rules in natural deduction.

The form of the above example should look somewhat familiar. Just as in the truth tree system, we number the statements and include a justification for every line. In this respect, the two systems are very similar. They diverge, however, in two important ways. For one, the natural deduction system also has no branching rules. More importantly though, within a natural deduction system, we must frequently make *sub-derivations*; there is no parallel for this in the other system.

Sub-derivations are like proofs within proofs. They begin with a premise and end with a statement derived from the premise. This premise is not a premise of the original argument — we have no reason to believe it is true. Rather, we simply *assume* its truth for the duration of the sub-derivation, and correspondingly its we write "assumption" as its justification. To show that we are currently making a sub-derivation, we draw a box around it, keeping it separate from the rest of the proof. As we have just assumed this premise to be true — as opposed to proving it to be true — we cannot use any of the lines of the sub-derivation outside the box. While we are within the sub-derivation itself, though, we can use statements that preceded it as long as they themselves are not closed within other sub-derivations.

Both the rules for negation use sub-derivations:

Negation Elimination or Negation Out (~O)

| ~α // assumption
| β
| ~β

∴ α

Negation Introduction or Negation In (~I)

| α // assumption
| β
| ~β

∴ ~α

These two rules illustrate why we cannot refer treat the assumption of the sub-derivation as true outside of the box. The reasoning behind them is as follows: if we assume the truth of a statement, say α, and from it — given the premises of the argument — we can deduce a contradiction, then α cannot possibly be true. Once we have deduced the contradiction, the sub-derivation ends so we close the box. But as we have shown that assuming α leads to a contradiction, we know that α cannot be true, and so we are licensed to conclude ~α. Our deduction may well continue, but we cannot access the lines within the box anymore. The reason for this is intuitively obvious — anything we deduced from α was deduced from a false premise and so is completely unreliable. If we could access the lines of the sub-derivation, then as we have β and ~β on separate line, we could deduce β & ~β by &I, and clearly we should not be able to do this.

The Out rule for v involves two sub-derivations.

Constructive Dilemma or Disjunction Out (vO)

α v β

| α // assumption
| γ

| β // assumption
| γ

∴ γ

The reasoning behind this is that if we are given α v β, then we know at least one of them must be true. If γ follows from both, then as we know at least one is true, we may safely conclude γ. As before, though, this is the *only* conclusion we can draw from the sub-derivations — none of the individual lines within them can be accessed once the boxes are closed.

The In rule for v is quite straightforward; if we know that α is true, then we must that — no matter the truth of β — α v β must be true.

Addition or Disjunction In (vI)

α
∴ α v β

or

α
∴ β v α

The Out rule for conditional is known as Modus Ponens. Given α ⊃ β, we know that if the antecedent is true, then the conclusion must be true, and so we can derive β if we are given α.

Modus Ponens (MP)

$\alpha \supset \beta$
α
$\therefore \beta$

The In rule for conditionals (Conditional Proof) uses a sub-derivation. If want to prove $\alpha \supset \beta$, then we begin by assuming α. We then try to derive β. If β follows from the truth of α, we can conclude that if β, then α — i.e. $\alpha \supset \beta$. If α is actually false, the conditional is true as its antecedent is false. If however α is true, then we have shown that β must then be true, and so the conditional is true as the conclusion is true.

Conditional Proof (CP)

| α //assumption
| β

$\therefore \alpha \supset \beta$

Derived Rules

The following three rules are not necessary for our natural deduction system, as their conclusions can be derived from their premises using the rules given above. They occur frequently though, and it saves time if we can refer directly to them.

The first rule acts like a second Out rule for conditional.

Modus Tollens (MT)

$\alpha \supset \beta$
$\sim\beta$
$\therefore \sim\alpha$

Modus Tollens is quite intuitive — we know that given $\alpha \supset \beta$, if the conclusion is false, then the antecedent must also be false. It can be derived easily from ~I and Modus Ponens as shown below.

1) $\alpha \supset \beta$
2) $\sim\beta$

3)	| α	Assumption
4)	| β	3, 1, MP
5)	| $\sim\beta$	2, R

6) $\sim\alpha$ 3-5, ~I

Hypothetical Syllogism (HS)

$\alpha \supset \beta$
$\beta \supset \gamma$
$\therefore \alpha \supset \gamma$

The Hypothetical Syllogism can be derived as follows:

1) $\alpha \supset \beta$		premise
2) $\beta \supset \gamma$		premise
3)	α	assumption
4)	β	3, 1 MP
5)	γ	4, 2 MP
6) $\alpha \supset \gamma$		3-5 CP

If we are able to reference it directly though, we can go straight from 1 and 2 to $\alpha \supset \gamma$ by HS.

Disjunctive Syllogism (DS)

α v β
$\sim\alpha$
$\therefore \beta$

or

α v β
$\sim\beta$
$\therefore \alpha$

The proof of the Disjunctive Syllogism is somewhat more complicated. To prove it, we must have a sub-derivation within a sub-derivation. This is perfectly legal, as long as we pay attention to the *scope* of a sub-derivation. Essentially we are within the scope of a sub-derivation as long as the box around it has not closed. Thus a sub-derivation within a sub-derivation can access lines in the outer sub-derivation, but not *vice-versa*.

1) α v β premise

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2) ~α                      premise

3)      | α                assumption
4)      |    
5)      |    | ~β          assumption
6)      |    | ~α          2, R
7)      |    | α           3, γ           //Note how we can access line three because we are
8)      |    |                              still within the scope of the outer sub-derivation.
9)      | β                5-7, ~O


10)     | β                assumption
11)     | β                10, R

12) β                      3-9, 10-11, vO
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Replacement Rules

In addition to the rules above, our natural deduction system makes use of some additional rules, which allow us to replace statements with other ones that are logically equivalent to it. The connective '::' here indicates that the statements on opposite sides of it are logically equivalent, and may be substituted for one another. Feel free to prove the validity of the following substitutions with truth tables or truth trees, or even natural deduction!

Commutation (Com)

α & β :: β & α
α v β :: β v α

Association (Assoc)

α & (β & γ) :: (α & β) & γ
α v (β v γ) :: (α v β) v γ

Implication (Impl)

α ⊃ β :: ~α v β

Double Negation (DN)

α :: ~~α

De Morgan (DeM)

~(α & β) :: ~α v ~β
~(α v β) :: ~α & ~β

Redundancy (Red)

α :: α & α
α :: α v α

Contraposition (Con)

α ⊃ β :: ~β ⊃ ~α

Exportation (Exp)

α ⊃ (β ⊃ γ) :: (α & β) ⊃ γ

Distribution (Dist)

α & (β v γ) :: (α & β) v (α & γ)
α v (β & γ) :: (α v β) & (α v γ)

See #9 below for a proof of Distribution.

Examples

All the following examples are taken from *Meaning and Argument*, chapters 4-7.

1) Chapter 4, exercise 2, # 1

1)	K & J & N	premise	
2)	~(K & M)	premise	We want to prove ~(N & M) from these premises.
3)	~K v ~M	2, DeM	
4)	│ ~K	assumption	
5)	│ K	1, &O	
6)	K	4-5, ~O	
7)	~M	6, 3, DS	
8)	~N v ~M	7, vI	
9)	~(N & M)	8, DeM	

2) Chapter 6, exercise 2, #3

1) ~(U v J)	premise	We want to prove ~(U & C).
2) ~U & ~J	1, DeM	
3) ~U	2, &O	
4) ~U v ~C	3, vI	
5) ~(U & C)	4, DeM	

3) Chapter 7, exercise 3, #1

1) P ⊃ R	premise	
2) R ⊃ ~S	premise	
3) A ⊃ S	premise	We want to prove P ⊃ ~A
4) P ⊃ ~S	1, 2, HS	
5) | P	assumption	
6) | ~S	5, 4 MP	
7) | ~A	6, 3, MT	
8) P ⊃ ~A	5-7, CP	

4) same page, #2

1)M	premise	
2) L ⊃ ~M	premise	
3) M ⊃ U	premise	
4) U ⊃ ~L	premise	We want to prove ~L
5) M ⊃ ~L	3, 4, HS	
6) ~L	5, 1 MP	

(Can you find an even quicker proof for the one above?)

5) same page, # 4

1) V ⊃ (B ⊃ T)	premise	
2) V	premise	
3) B	premise	
4) ~W ⊃ ~T	premise	
5) ~C ⊃ ~W	premise	We want to prove C
6) B ⊃ T	1, 2, MP	
7) T	6, 3, MP	

8) W	7, 4 MT	
9) C	8, 5 MT	

6) same page, #12

1) ~(C & P)	premise	
2) ~C ⊃ A	premise	
3) ~A	premise	We want to prove ~P
4) C	3, 2, MT	
5) ~C v ~P	1, DeM	
6) ~P	5, 4, DS	

7) same page, # 15

1) P ⊃ ~(C v S)	premise	
2) ~~C & ~~S	premise	We want to prove ~P
3) ~~C	2, &O	
4) ~~S	2, &O	//Notice here that we cannot apply DN (Double Negation) to the parts of 2. We must first break it down into its components
5) C	3, DN	
6) S	4, DN	
7) C v S	5, vI	
8) ~P	7, 1 MT	

8) same page, #25

1) ~T ⊃ ~(W v R)		premise	
2) T ⊃ ~(W v R)		premise	We want to prove ~(W v R)
3)	W v R	assumption	
4)	T	3, 1 MT	
5)	~T	3, 2 MT	
6) ~ (W v R)		3-5 ~I	

9) Prove the Replacement Rule of Distribution:

α & (β v γ) :: (α & β) v (α & γ)
α v (β & γ) :: (α v β) & (α v γ)

To prove this rule, we need four proofs. As we need to show that the two statements in each pair are logically equivalent, we have to prove that the first entails the second, and then that the second entails the first (see Meaning and Argument section 8.6.3).

Derivation of α & (β v γ) from ((α & β) v (α & γ))

1)	((α & β) v (α & γ))	Premise
2)	| α & β	Assumption
3)	| α	2, &O
4)	| α & γ	Assumption
5)	| α	4, &O
6)	α	1, 2-3, 4-5 vO
7)	| α & β	Assumption
8)	| β	7, &O
9)	| β v γ	8, vI
10)	| α & γ	Assumption
11)	| γ	10, &O
12)	| β v γ	11, vI
13)	β v γ	1, 7-9, 10-12, vO
14)	α & (β v γ)	6, 13, &I

Derivation of ((α & β) v (α & γ)) from α & (β v γ)

1)	α & (β v γ)	Premise
2)	α	1, &O
3)	β v γ	1, &O
4)	| ~((α & β) v (α & γ))	Assumption
5)	| ~(α & β) & ~ (α & γ)	4, De Morgan's (DeM)
6)	| ~(α & β)	5, &O
7)	| ~(α & γ)	5, &O
8)	| ~α v ~ β	6, DeM
9)	| ~α v ~γ	7, DeM
10)	| ~β	2, 8, Disjunctive Syllogism (DS)
11)	| ~γ	2, 9, DS
12)	| ~β & ~γ	10, 11, &I

13) | ~(β v γ) 12, DeM
14) | β v γ 3, Reiteration (R)

15) ((α & β) v (α & γ)) 4-14, ~O

Derivation of ((α v β) & (α v γ)) from α v (β & γ)

1) α v (β & γ) Premise

2) | ~((α v β) & (α v γ)) Assumption
3) | ~(α v β) v ~(α v γ) 2, DeM

4) | | ~(α v β) Assumption
5) | | ~α & ~β 4, DeM
6) | | ~α 5, &O
7) | | ~β 5, &O
8) | | ~β v ~γ 7, vI
9) | | ~α & (~β v ~γ) 6, 8, &I

10) | | ~(α v γ) Assumption
11) | | ~α & ~γ 7, DeM
12) | | ~α 11, &O
13) | | ~γ 11, &O
14) | | ~β v ~γ 13, vI
15) | | ~α & (~β v ~γ) 12, 14 &I

16) | ~α & (~β v ~γ) 3, 4-9, 10-15, vo
17) | ~α 16, &O
18) | ~β v ~γ 16 &O
19) | ~(β & γ) 18, DeM
20) | ~α & ~ (β & γ) 17, 19 &I
21) | ~(α v (β & γ)) 20, DeM
22) | α v (β & γ) 1, R

23) ((α v β) & (α v γ)) 2-22, ~O

Derivation of α v (β & γ) from ((α v β) & (α v γ))

1) (α v β) & (α v γ) Assumption
2) α v β 1, &O
3) α v γ 1, &O

4)	$\sim(\alpha \vee (\beta \mathbin{\&} \gamma))$	Assumption
5)	$\sim \alpha \mathbin{\&} \sim(\beta \mathbin{\&} \gamma)$	4, DeM
6)	$\sim\alpha$	5, &O
7)	$\sim(\beta \mathbin{\&} \gamma)$	5, &O
8)	β	6, 2 HS
9)	γ	6, 3 HS
10)	$\beta \mathbin{\&} \gamma$	8, 9 &I
11)	$\sim(\beta \mathbin{\&} \gamma)$	7, R

12) $\alpha \vee (\beta \mathbin{\&} \gamma)$ 4-11, ~O