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Theoretical note

A catalog of Boolean concepts

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Abstract

Boolean concepts are concepts whose membership is determined by a Boolean function, such as that expressed by a formula of propositional logic. Certain Boolean concepts have been much studied in the psychological literature, in particular with regard to their ease of learning. But research attention has been somewhat uneven, with a great deal of attention paid to certain concepts and little to others, in part because of the unavailability of a comprehensive catalog. This paper gives a complete classification of Boolean concepts up to congruence (isomorphism of logical form). Tables give complete details of all concepts determined by up to four Boolean variables. For each concept type, the tables give a canonic logical expression, an approximately minimal logical expression, the Boolean complexity (length of the minimal expression), the number of distinct Boolean concepts of that type, and a pictorial depiction of the concept as a set of vertices in Boolean *D*-space. Some psychological properties of Boolean concepts are also discussed.

1. Boolean concepts

A *Boolean variable* is a variable that can take one of two distinct values, e.g., 0 or 1, often thought of as "truth values" with 1 meaning "true" and 0 meaning "false" (Boole, 1854/1958). *Boolean D-space* is the space created by crossing *D* Boolean variables. Such a space can be conveniently thought of as a *D*-dimensional cube or hypercube, with each vertex corresponding to one possible combination of truth values for each of the *D* variables, that is, a *D*-dimensional Boolean object.

A *Boolean function* is a function mapping Boolean *D*-space to $\{0, 1\}$, that is, assigning a particular subset of the vertices of *D*-space to "true" and the rest to "false." Such a function can be conveniently depicted as a set of vertices in *D*-space, namely those designated "true"; these are sometimes called the *positive* examples or vertices, and the rest the *negative*. Boolean functions are specified completely by the set of vertices they assign as positive (and thus negative); two functions with the same positive (or negative) vertices are the same function. An excellent mathematical survey of Boolean functions can be found in Wegener (1987) (see also Paterson, 1992).

A *Boolean concept* is simply a Boolean function thought of this way as a set of vertices in Boolean *D*-space. The terminology reflects the fact that such a concept picks out a specific set of objects from the space of possible objects (i.e., picks out a particular subset of Boolean *D*-space), in much the same way that the concept "dog" picks out a particular subset from the space of possible entities, namely, those that are dogs.

The space of possible Boolean concepts comprises a great variety of structures and patterns that have yet to be investigated in detail. The purpose of this paper is to catalog and classify the Boolean concepts of up to four variables, as a reference for the benefit of other researchers.

1.1. Boolean concepts in psychology

Interest in the psychology of Boolean concepts began in the 1950s with the realization that distinct Boolean functions differed in various psychologically important ways, such as the ease with which they were learned from examples (Bruner, Goodnow, & Austin, 1956). An enormous flow of research ensued during the following two decades, some of which is summarized below. During this period psychological properties of various Boolean concepts were studied from a variety of perspectives, including comparisons across cultures

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(Ciborowski & Cole, 1972) and across species (Wells & Deffenbacher, 1967). However interest in Boolean concepts began to diminish in the 1970s as evidence accumulated that membership in psychological categories exhibits gradations—variations in the degree of category membership—not exhibited by Boolean concepts (Posner & Keele, 1968; Rosch, 1973; Armstrong, Gleitman, & Gleitman, 1983). Modern categorization theories, mostly either "prototype" or "exemplar" models, emphasize the graded nature of membership in psychological categories.

However, the evidence in favor of a graded view of concepts relates only to the way concepts are represented mentally. Thus while there is much evidence that concepts are not represented *qua* Boolean functions, there is no evidence that the mental representation does not in some way depend on the logical form of the target concept. Indeed, it obviously does depend on it; for any reasonable theory of categorization, including prototype and exemplar models, the ease with which a concept will be represented or stored will depend in part on the internal structure of the concept, which for Boolean features means the logical form of the Boolean function that specifies the positive examples (or some other specially designated set). That is, it is to be expected that the way a set of objects will be mentally apprehended as a unitary concept will depend substantially on the internal logical structure of the set, and this sort of dependency is still a standard part of every modern theory of concepts. Hence the enumeration and understanding of the range of possible such structures—the topic of the current article—is still a critical part of the study of the mental representation of concepts. It is the range of possible *objects* of concept representation, though probably not the range of possible concept *representations*. In fact, despite the loss of interest in Boolean functions as models of concept representation, Boolean concepts are still what subjects are actually asked to learn in most modern studies of categorization (the major exception to this generalization are studies of, e.g., decision-bound categorization, in which the variables are usually continuous).

A complete survey of the space of Boolean concepts is especially important because of the somewhat uneven way in which this space has historically been canvassed in studies of psychological mechanisms. The great quantity of research in the 1960s focused on concepts defined by only two features, which due to their relative simplicity are in many ways atypical. In retrospect, this seems to have led to some erroneous generalizations about psychological mechanisms (Feldman, 2000). More recently, many studies have repeatedly used the same few concept types, in part to ease comparisons among models. However, Smith and Minda (2000) have argued that this has led to an overestimate of the evidence in favor of particular models, due to certain aspects of the structure of those particular concepts. Hence the need for a more complete and level "playing field" is urgent.

Aside from category learning, Boolean functions also relate to several other topics of potentially great interest to psychologists and cognitive scientists. Reasoning, in particular deductive (logically certain) reasoning (Johnson-Laird, 1983), intrinsically relates to Boolean functions, in that each such function can be thought of as a different way that the truth or falsity of a conclusion can depend functionally on the truth or falsity of a set of premises. The study of abstract neural networks also relates to Boolean concepts, in that each Boolean function corresponds to a possible input–output mapping for an abstract neural net of the McCulloch-Pitts type (McCulloch & Pitts, 1943). Finally, philosophical interest in the compositionality of human concepts—that is, the idea that the meaning of a concept must be functionally dependent on the meaning of its constituent elements (Fodor, 1994)—also intrinsically relates to Boolean concepts, each of which specifies the details of one such compositional function.

1.2. Propositional formulae

A *propositional formula* or *Boolean formula* is a string of symbols constructed from Boolean variables (written a, b, etc.) using logical connectives (e.g., $\wedge, \vee, \neg$) in the familiar manner. Various choices of connectives are possible, including the usual choice $\{\wedge, \vee, \neg\}$ which I will use below; the set chosen is called the *basis*. I will adopt the notation usually favored by mathematicians (and Boole himself; Boole, 1854/1958) in which $a \wedge b$ is written ab, $a \vee b$ is written $a + b$, and $\neg a$ is written a'. Hence a', $a + b'$, $a(b + c)'$ are all propositional formulae. Each positive or negative variable is called a *literal*.

The set of truth values that satisfy a given propositional formula defines a set of vertices in Boolean D-space, and thus in effect a specific Boolean concept. Any Boolean concept can in fact be expressed exactly by an infinite number of distinct propositional formulae. Formulae that express the same Boolean function or concept are called *equivalent*.

A *disjunctive normal formula* or DNF is a formula that is a disjunction of conjunctions of literals, such as $ab + cd$. Every propositional formula is equivalent to at least one DNF, a fact that becomes obvious when one considers that there exists a DNF in which each conjunctive clause picks out as positive one vertex in the Boolean D-space representation of the corresponding Boolean concept. For example the DNF $abc' + a'b'c$ explicitly picks out two vertices from Boolean 3-space, namely abc' and $a'b'c$. For each concept, this type of DNF is unique (up to the order of clauses and the order of variables within each clause, which are arbitrary).

Henceforth this DNF will be referred to as "the" DNF for the concept.

It is very convenient to refer to Boolean concepts by using a corresponding formula, in particular its DNF, as I will do below. However, it is very important to keep in mind that a propositional formula and the Boolean concept it picks out are *not* the same thing. For example the specification of a Boolean concept does not involve any particular choice of basis. Rather the concept is defined completely by its pattern of vertices in D-space, which does not in any way depend on how it might be expressed as a formula. This fact is crucial in understanding the classification given below.

1.3. Congruence

Some Boolean concepts, while not equivalent to each other, seem intuitively to refer to the same "type" of concept. For example, the concepts $a + b'$ and $a' + b$ are not equivalent, but seem to be of essentially the same kind, in the sense that they would be equivalent if we simply switched the labels a and b, which are after all arbitrary. In some contexts, the polarity of each variable might also be arbitrary, such as if a means "square" and a' "triangle," in which case neither value seems to have any special claim to the label "true." Hence the concepts a and a' might be regarded as of the same type, although, again, they are certainly not equivalent.

I will call this kind of similarity among concepts *congruence*; two concepts are called *congruent* if the two may be made equivalent by a consistent reassignment of the labels and polarities of the variables.[1] Intuitively, congruence between two concepts means that their images in D-space (that is, the set of positive vertices) are rigid rotations or mirror reflections of each other. This notion of congruence seems to have been first introduced by Aiken and his colleagues (Aiken & the Staff of the Computation Laboratory at Harvard University, 1951), and subsequently became common in the literature on the theory of switching circuits. It was introduced into psychology by Shepard, Hovland and Jenkins (1961), and used more recently in Feldman (2000).

Of course, not all concepts for a given number of features D are congruent, and so the next question to ask is: what are the possible distinct types or equivalence classes? Clearly, for two concepts to be congruent, they must have the same number P of positive vertices. Hence the enumeration of equivalence classes necessarily depends on D and P. Shepard et al. (1961) pointed out that for $D = 3$ and $P = 4$, there are six basic classes, which they denoted using Roman numerals I–VI. This kind of typology is crucial for the study of psychological properties that depend on the *logical form* of concepts, because all concepts within the type have essentially the same logical form—they are the same when we disregard superficial details about how properties are labeled—while concepts of different types have qualitatively different logical forms.

In psychological studies of concepts, concepts have usually been studied modulo congruence; that is, psychological properties are associated with an entire class of congruent concepts rather than one specific concept. A prominent example is Shepard et al.'s set of six types (Shepard et al., 1961). The many subsequent studies of these types (e.g., Kruschke, 1992; Nosofsky, Gluck, Palmeri, McKinley, & Glauthier, 1994a; Nosofsky, Palmeri, & McKinley, 1994b) have consistently respected the typology up to congruence. There have been occasional exceptions, however, especially in the 1960s. Implicational concepts (e.g., $a \rightarrow b$) have occasionally been distinguished from disjunctive concepts (e.g., $a + b$), to which they are congruent (because $a \rightarrow b$ means $a' + b$). Similarly *affirmation* (e.g., a) has sometimes been distinguished from *negation* (a'), which again are obviously congruent. Such distinctions necessarily entail that positive values of features are somehow distinguishable from negative values, i.e., that features have intrinsic polarity. The classification into congruence classes below presumes this is never the situation.

2. A complete classification of Boolean concepts

As mentioned above, Shepard et al. (1961) used an explicit typology for $D = 3, P = 4$. A complete typology for all $D \leqslant 4$ and all P was given in Aiken et al.'s remarkable 1951 monograph, but in somewhat antiquated and difficult notation (directed at early designers of vacuum-tube switching circuits). The main purpose of this paper is to present the typology in more modern notation, organized by values of D and P, and giving certain additional information about each concept, most importantly (i) simplified expressions in standard logical notation, and (ii) visual representations as sets of vertices in Boolean D-space.

2.1. Notation

As discussed above, the possible types necessarily depend on both D and P, because two concepts with different values of either D or P cannot be congruent to each other. I will denote the family of types for D and P as $D[P]$, and the distinct types or cases in it by numerals subscripted by the family name, i.e., $C_{D[P]}$, with C a number running from 1 to $|D[P]|$. Because Roman

[1] Terminology for this relationship in the literature is inconsistent. In some sources such concepts are called *isomorphic*, but this term is somewhat nonspecific, Harrison (1965) refers to such concepts as *equivalent*, but this conflicts with the logicians' stricter use of this term, also used in the current paper, to refer to formulae that pick out exactly the same Boolean function.

numerals become unwieldy with the large families enumerated below, in this catalog I will use bold Arabic numerals for case labels. Under this system Shepard et al.'s family is 3[4], with $|3[4]| = 6$, and the six cases are $\mathbf{1}_{3[4]}$ through $\mathbf{6}_{3[4]}$. The numbering of the cases within each family is arbitrary;[2] one of the goals of this paper is to establish conventional labels.

Feldman (2000) used explicit typologies for the families 3[2], 3[3], 3[4], 4[2], 4[3], and 4[4], although due to space limitations details of the typologies were given in that paper only for type 3[3] and 3[4] (Shepard et al.'s family). The tables below give explicit typologies for $D = 2, 3$ and 4, and for all nontrivial values of P.

2.2. Complementary concepts

Every concept ϕ has a complementary concept ϕ' in which the assignment of vertices is reversed (positive becoming negative, negative positive). Such a concept is necessarily noncongruent to ϕ (unless $P = 2^{D-1}$, in which case the two are mirror images of each other and necessarily congruent). Nevertheless ϕ and ϕ' have in a sense the same structure, except inverted, and enumerating them separately is superfluous. Hence for economy the catalog below includes only cases where $P \leqslant 2^{D-1}$, bearing in mind the existence of "twin" cases with $P > 2^{D-1}$.

The identification of concepts with their complements, coupled with the definition of congruence, means that the classification collapses together some concepts often thought of as distinct. As mentioned, the concepts a and a', respectively affirmation and negation, are congruent, and thus represent the same case in the tables below ($\mathbf{1}_{2[2]}$). Similarly, the concepts ab and $a + b$, respectively conjunction and disjunction, are complementary (that is, the complement of one is congruent to the other), and so again represent the same case ($\mathbf{1}_{2[1]}$).

2.3. Organization of the catalog

The catalog given below is a complete classification of Boolean concepts (with $D \leqslant 4$) up to congruence (omitting only those concepts with $P = 0$, of which there is exactly one for each D). For each case the table gives D, P, the DNF,[3] and an illustration of the concept as a set of vertices in Boolean D-space. The tables also give several other kinds of information that will be explained below, including the population (N) of each case, a minimal equivalent propositional formula, and the Boolean complexity of the concept.

[2] More strictly, the concepts are labeled in lexicographic order by their first member (assuming $a < b < \cdots$ and $a' < a$), with the sole exception of 3[4], whose members have been given Shepard et al.'s numbers in deference to convention.

[3] More precisely, the table gives *one* example of each concept in DNF form, namely the lexicographically earliest one.

Table 1 gives the number of cases in all the families with $D \leqslant 4$ (as well as 5[1], 5[2], and 5[3], which are not detailed in the tables). The way that family population varies as D and P are varied is, in some ways, surprisingly idiosyncratic and seemingly unpredictable. Harrison (1965) gives a detailed discussion of and expressions for these numbers. Certain patterns are immediately obvious. For all D, $|D[1]| \equiv 1$, because all single vertices of D-space are equivalent after rotation. Similarly, $|D[2]| \equiv D$, because pairs of vertices differ only in how many edges separate them. Above $P = 2$, obvious patterns diminish; see Harrison (1965) for a more detailed discussion.

It is worth reiterating that the intrinsic structure of a Boolean concept does not depend in any way on the propositional formula that may be used to represent it. Hence the catalog given here (and its infinite extension), because it lists all qualitatively distinct basic structures, is universal; it is unrelated to any particular propositional representation or basis. Hence it constitutes the natural space in which any investigation of Boolean concepts naturally resides.[4]

2.4. Minimal formulae and Boolean complexity

As mentioned above, each Boolean concept can be described by an infinite number of distinct propositional formulae. The shortest such formula is called the *minimal equivalent formula*. In many cases the properties of a particular concept can be appreciated most easily by inspecting its minimal formula, as will be seen below.

The length of the minimal formula (usually given in literals), called the *Boolean complexity*, is a measure of the inherent logical complexity or incompressibility of the concept (Givone, 1970). Boolean complexity has properties paralleling those of Kolmogorov complexity (Kolmogorov, 1965; Solomonoff, 1964; Chaitin, 1966; Li & Vitányi, 1997). In particular (a) Boolean complexity takes on the maximal value of DP for concepts that cannot be compressed at all; that is, for which no representation is more efficient than a verbatim DNF rendition, and (b) Boolean complexity is universal, up to a multiplicative constant, with respect to choice of basis (Wegener, 1987).

Unfortunately, the computation of minimal equivalent formulae is intractable (Garey & Johnson, 1979), but approximately minimal expressions can be found rapidly via factoring and other heuristic techniques. The minimal formulae given in the tables below are derived from such heuristics and are thus *not* necessarily globally minimal; the complexity values should be

[4] That is, unless the conditions assumed in the definition of congruence are not met. For example if the assignment of true and false is *not* arbitrary, then a and a' should no longer be regarded as congruent forms, and the resulting classification would change.

Table 1
Populations of families in the Boolean $D[P]$ hierarchy. Each cell gives the number of concepts $|D[P]|$ in the family $D[P]$

	$\lvert D[P]\rvert$	P: 1	2	3	4	5	6	7	8
D	1	1							
	2	1	2						
	3	1	3	3	6				
	4	1	4	6	19	27	50	56	74
	5	1	5	10					

regarded as approximations, or, more strictly, upper bounds.

2.5. Populations of cases

Every Boolean concept is congruent to one of the cases $C_{D[P]}$ in the tables. For a given family $D[P]$, however, the distribution of concepts among the various cases is not uniform; some types occur more often than others. The tables below give the "population" (N) for each case. These numbers depend in a subtle way on the degree of structural symmetry within each concept.

The populations obey certain obvious equalities. The numbers of distinct concepts $N_{C_{D[P]}}$ in each concept type $C_{D[P]}$ must add up to the total number of concepts in the family $D[P]$. This latter quantity is simply the number of ways the 2^D objects may be taken P at a time. Hence we have the following relation:

$$\frac{2^D!}{(2^D-P)!P!} = \sum_{C \in D[P]} N_{C_{D[P]}}. \tag{1}$$

Summing over all families, the total number of concepts of dimension D must be the total in all constituent families $D[P]$. Note however that because families with $P < 2^{D-1}$ have complementary "twin" families that do not appear in the tables, the summation doubles their counts, hence:

$$2^{2^D} = 2 \sum_{P<2^{D-1}} N_{C_{D[P]}} + \sum_{P=2^{D-1}} N_{C_{D[P]}}. \tag{2}$$

Note that this summation includes the trivial cases with $P = 0$ and $P = 2^D$ (of which there are each one for every D), which are not included in the tables.

2.6. Remarks

As D and P grow, the number of distinct cases grows rapidly. Hence although the aim of the current paper is to give details of these cases, for the later families a coarser and more informative classification seems desirable. I do not make a systematic attempt at one here. However in certain instances, cases in distinct families have affinities that deserve notice.

Several of the concepts in the tables, or their complements, have conventional names. As mentioned above, $\mathbf{1}_{2[2]}$ is usually called affirmation (or negation); the reason is clear when one looks at its minimal formula, a', which simply affirms (or negates) a single variable (i.e., all members of such a concept share a common feature). Hence by extension, any concept of dimension D and Boolean complexity 1 constitutes a version of affirmation, i.e., D-dimensional affirmation.[5] Because affirmation divides D-space exactly in half, all and only families with $P = 2^{D-1}$, such as 2[2], 3[4], 4[8], etc., contain a version of D-affirmation.

Similarly, the name *conjunction* (*disjunction*) has in the psychological literature usually been associated with $\mathbf{1}_{2[1]}$. But by extension the term D-conjunction would fit any concept whose minimal formula was the conjunction of D literals, thus bearing complexity D. In fact for every D, D-conjunction is the sole member of the family $D[1]$.

$\mathbf{2}_{2[2]}$ is usually called *exclusive-or* or *biconditional* (the latter because it is equivalent to $(a \rightarrow b)(b \rightarrow a)$). This function has the property that any change in the value of exactly one variable (i.e., a change of "parity") always leads to a change in the function's value. Hence it and its higher-dimensional analogs (e.g., $\mathbf{6}_{3[4]}$, $\mathbf{74}_{4[8]}$) are sometimes referred to as versions of the *parity function*. As with affirmation, a version of the parity function exists only in families with $P = 2^{D-1}$. Parity is extremely incompressible (see Schöning & Pruim, 1998), and thus for a given value of D usually has the highest Boolean complexity (only slightly below the absolute complexity limit DP).

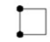
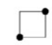
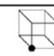
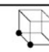
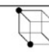
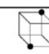
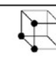
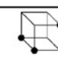
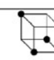
3. Psychological properties

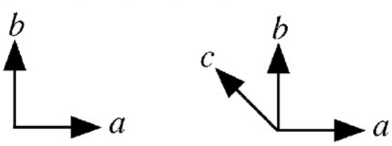
The remainder of my comments focus on what is known about psychological properties of various concept types.

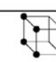
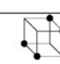
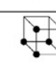
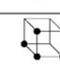
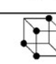
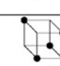
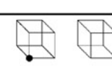
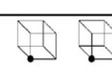
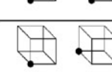
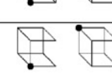
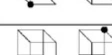
Several studies have employed one or another type in studies of generalization. Such studies have typically trained subjects on examples from a single fixed concept; interest then focuses on the degree to which novel

[5] In fact, it would be fair to regard such concepts as in a sense congruent, in that their minimal formulae are congruent; but it is clearer to regard them as distinct concepts in that they reside in different families.

Table 2

Case	*D*	*P*	*N*	*DNF*	*Minimal formula*	*Complexity*	*Illustration*
2[1] $\mathbf{1}_{2[1]}$	2	1	4	$a'b'$	$a'b'$	2	
2[2] $\mathbf{1}_{2[2]}$	2	2	4	$a'b' + a'b$	a'	1	
$\mathbf{2}_{2[2]}$	2	2	2	$a'b' + ab$	$a'b' + ab$	4	
3[1] $\mathbf{1}_{3[1]}$	3	1	8	$a'b'c'$	$a'b'c'$	3	
3[2] $\mathbf{1}_{3[2]}$	3	2	12	$a'b'c' + a'b'c$	$a'b'$	2	
$\mathbf{2}_{3[2]}$	3	2	12	$a'b'c' + a'bc$	$a'(b'c' + bc)$	5	
$\mathbf{3}_{3[2]}$	3	2	4	$a'b'c' + abc$	$a'b'c' + abc$	6	
3[3] $\mathbf{1}_{3[3]}$	3	3	24	$a'b'c' + a'b'c + a'bc'$	$a'(bc)'$	3	
$\mathbf{2}_{3[3]}$	3	3	24	$a'b'c' + a'b'c + abc'$	$a'b' + abc'$	5	
$\mathbf{3}_{3[3]}$	3	3	8	$a'b'c' + a'bc + ab'c$	$a'(b'c' + bc) + ab'c$	8	



Case	*D*	*P*	*N*	*DNF*	*Minimal formula*	*Complexity*	*Illustration*
3[4] $\mathbf{1}_{3[4]}$	3	4	6	$a'b'c' + a'b'c + a'bc' + a'bc$	a'	1	
$\mathbf{2}_{3[4]}$	3	4	6	$a'b'c' + a'b'c + abc' + abc$	$ab + a'b'$	4	
$\mathbf{3}_{3[4]}$	3	4	24	$a'b'c' + a'b'c + a'bc' + ab'c$	$a'(bc)' + ab'c$	6	
$\mathbf{4}_{3[4]}$	3	4	8	$a'b'c' + a'b'c + a'bc' + ab'c'$	$a'(bc)' + ab'c'$	6	
$\mathbf{5}_{3[4]}$	3	4	24	$a'b'c' + a'b'c + a'bc' + abc$	$a'(bc)' + abc$	6	
$\mathbf{6}_{3[4]}$	3	4	2	$a'b'c' + a'bc + ab'c + abc'$	$a(b'c + bc') + a'(b'c' + bc)$	10	
4[1] $\mathbf{1}_{4[1]}$	4	1	16	$a'b'c'd'$	$a'b'c'd'$	4	
4[2] $\mathbf{1}_{4[2]}$	4	2	32	$a'b'c'd' + a'b'c'd$	$a'b'c'$	3	
$\mathbf{2}_{4[2]}$	4	2	48	$a'b'c'd' + a'b'cd$	$a'b'(c'd' + cd)$	6	
$\mathbf{3}_{4[2]}$	4	2	32	$a'b'c'd' + a'bcd$	$a'(b'c'd' + bcd)$	7	
$\mathbf{4}_{4[2]}$	4	2	8	$a'b'c'd' + abcd$	$a'b'c'd' + abcd$	8	

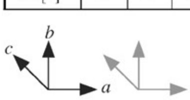


Table 2 (*continued*)

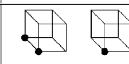
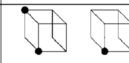
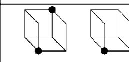
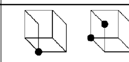
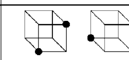
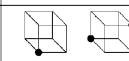
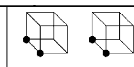
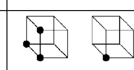
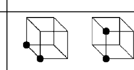
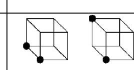
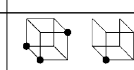
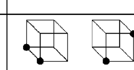
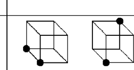
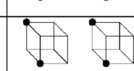
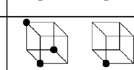
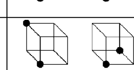
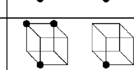
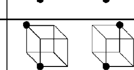
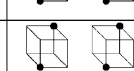
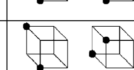
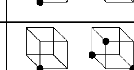
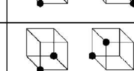
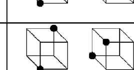
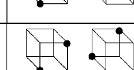
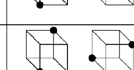
Case	*D*	*P*	*N*	*DNF*	*Minimal Formula*	*Complexity*	*Illustration*
4[3] $\mathbf{1}_{4[3]}$	4	3	96	$a'b'c'd' + a'b'c'd + a'b'cd'$	$a'b'(cd)'$	4	
$\mathbf{2}_{4[3]}$	4	3	192	$a'b'c'd' + a'b'c'd + a'bcd'$	$a'(b'c' + bcd')$	6	
$\mathbf{3}_{4[3]}$	4	3	64	$a'b'c'd' + a'b'c'd + abcd'$	$a'b'c' + abcd'$	7	
$\mathbf{4}_{4[3]}$	4	3	64	$a'b'c'd' + a'b'cd + a'bc'd$	$a'(b'(c'd' + cd) + bc'd)$	9	
$\mathbf{5}_{4[3]}$	4	3	48	$a'b'c'd' + a'b'cd + abc'd'$	$a'b'(c'd' + cd) + abc'd'$	10	
$\mathbf{6}_{4[3]}$	4	3	96	$a'b'c'd' + a'b'cd + abc'd$	$a'b'(c'd' + cd) + abc'd$	10	
4[4] $\mathbf{1}_{4[4]}$	4	4	24	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd$	$a'b'$	2	
$\mathbf{2}_{4[4]}$	4	4	64	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd'$	$a'(b'(cd)' + bc'd')$	7	
$\mathbf{3}_{4[4]}$	4	4	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd$	$a'(b'(cd)' + bdc')$	7	
$\mathbf{4}_{4[4]}$	4	4	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bcd$	$a'(b'(cd)' + bcd)$	7	
$\mathbf{5}_{4[4]}$	4	4	96	$a'b'c'd' + a'b'c'd + a'b'cd' + abc'd'$	$a'b'(cd)' + abc'd'$	8	
$\mathbf{6}_{4[4]}$	4	4	192	$a'b'c'd' + a'b'c'd + a'b'cd' + abc'd$	$a'b'(cd)' + abc'd$	8	
$\mathbf{7}_{4[4]}$	4	4	96	$a'b'c'd' + a'b'c'd + a'b'cd' + abcd$	$a'b'(cd)' + abcd$	8	
$\mathbf{8}_{4[4]}$	4	4	48	$a'b'c'd' + a'b'c'd + a'bcd' + a'bcd$	$a'(bc + b'c')$	5	
$\mathbf{9}_{4[4]}$	4	4	192	$a'b'c'd' + a'b'c'd + a'bcd' + ab'cd'$	$cd'(a'b + ab') + a'b'c'$	9	
$\mathbf{10}_{4[4]}$	4	4	192	$a'b'c'd' + a'b'c'd + a'bcd' + ab'cd$	$a'(b'c' + bcd') + ab'cd$	10	
$\mathbf{11}_{4[4]}$	4	4	96	$a'b'c'd' + a'b'c'd + a'bcd' + abcd'$	$bcd' + a'b'c'$	6	
$\mathbf{12}_{4[4]}$	4	4	192	$a'b'c'd' + a'b'c'd + a'bcd' + abcd$	$bc(a'd' + ad) + a'b'c'$	9	
$\mathbf{13}_{4[4]}$	4	4	16	$a'b'c'd' + a'b'c'd + abcd' + abcd$	$abc + a'b'c'$	6	
$\mathbf{14}_{4[4]}$	4	4	16	$a'b'c'd' + a'b'cd + a'bc'd + a'bcd'$	$a'(b(c'd + cd') + b'(c'd' + cd))$	11	
$\mathbf{15}_{4[4]}$	4	4	16	$a'b'c'd' + a'b'cd + a'bc'd + ab'c'd$	$a'(b'(c'd' + cd) + bc'd) + ab'c'd$	13	
$\mathbf{16}_{4[4]}$	4	4	96	$a'b'c'd' + a'b'cd + a'bc'd + ab'cd'$	$a'(b'(c'd' + cd) + bc'd) + ab'cd'$	13	
$\mathbf{17}_{4[4]}$	4	4	64	$a'b'c'd' + a'b'cd + a'bc'd + abcd'$	$a'(b'(c'd' + cd) + bc'd) + abcd'$	13	
$\mathbf{18}_{4[4]}$	4	4	12	$a'b'c'd' + a'b'cd + abc'd' + abcd$	$(c'd' + cd)\,(ab + a'b')$	8	
$\mathbf{19}_{4[4]}$	4	4	24	$a'b'c'd' + a'b'cd + abc'd + abcd'$	$ab\,(c'd + cd') + a'b'\,(c'd' + cd)$	12	

Table 2 (*continued*)

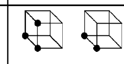
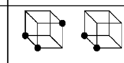
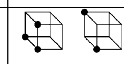
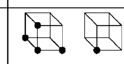
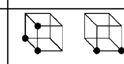
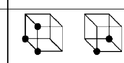
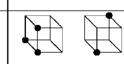
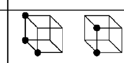
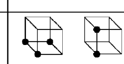
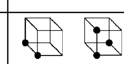
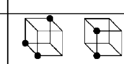
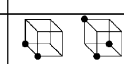
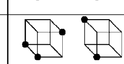
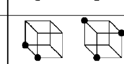
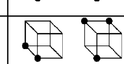
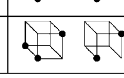
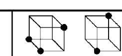
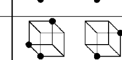
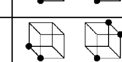
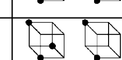
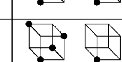
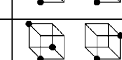
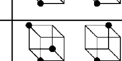
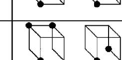
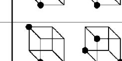
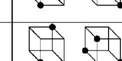
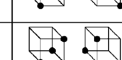
Case	*D*	*P*	*N*	*DNF*	*Minimal Formula*	*Complexity*	*Illustration*
4[5] $\mathbf{1}_{4[5]}$	4	5	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd'$	$a'(b' + bc'd')$	5	
$\mathbf{2}_{4[5]}$	4	5	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + abc'd'$	$a'b' + abc'd'$	6	
$\mathbf{3}_{4[5]}$	4	5	64	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + a'bcd$	$a'(b(cd' + c'd) + b'cd)'$	9	
$\mathbf{4}_{4[5]}$	4	5	16	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'c'd'$	$c'd'(a'b + ab') + a'b'(cd)'$	10	
$\mathbf{5}_{4[5]}$	4	5	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'c'd$	$c'(a'bd' + ab'd) + a'b'(cd)'$	11	
$\mathbf{6}_{4[5]}$	4	5	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'cd$	$b'(a'(cd)' + acd) + ba'c'd'$	11	
$\mathbf{7}_{4[5]}$	4	5	64	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + abcd$	$b(a'c'd' + acd) + a'b'(cd)'$	11	
$\mathbf{8}_{4[5]}$	4	5	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + a'bcd'$	$a'(cd + bc'd')'$	6	
$\mathbf{9}_{4[5]}$	4	5	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + ab'cd'$	$b'(a'(cd)' + acd') + bda'c'$	11	
$\mathbf{10}_{4[5]}$	4	5	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + ab'cd$	$d(a'bc' + ab'c) + a'b'(cd)'$	11	
$\mathbf{11}_{4[5]}$	4	5	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + abcd'$	$b(a'c'd + acd') + a'b'(cd)'$	11	
$\mathbf{12}_{4[5]}$	4	5	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bcd + ab'cd$	$cd(a'b + ab') + a'b'(cd)'$	10	
$\mathbf{13}_{4[5]}$	4	5	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bcd + abc'd'$	$b(a'cd + ac'd') + a'b'(cd)'$	11	
$\mathbf{14}_{4[5]}$	4	5	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bcd + abc'd$	$bd(a'c + ac') + a'b'(cd)'$	10	
$\mathbf{15}_{4[5]}$	4	5	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bcd + abcd$	$bcd + a'b'(cd)'$	7	
$\mathbf{16}_{4[5]}$	4	5	192	$a'b'c'd' + a'b'c'd + a'b'cd' + abc'd' + abc'd$	$abc' + a'b'(cd)'$	7	
$\mathbf{17}_{4[5]}$	4	5	96	$a'b'c'd' + a'b'c'd + a'b'cd' + abc'd' + abcd$	$ab(c'd' + cd) + a'b'(cd)'$	10	
$\mathbf{18}_{4[5]}$	4	5	96	$a'b'c'd' + a'b'c'd + a'b'cd' + abc'd + abcd'$	$ab(c'd + cd') + a'b'(cd)'$	10	
$\mathbf{19}_{4[5]}$	4	5	192	$a'b'c'd' + a'b'c'd + a'b'cd' + abc'd + abcd$	$abd + a'b'(cd)'$	7	
$\mathbf{20}_{4[5]}$	4	5	192	$a'b'c'd' + a'b'c'd + a'bcd' + a'bcd + ab'cd'$	$a'(bc + b'c') + ab'cd'$	9	
$\mathbf{21}_{4[5]}$	4	5	64	$a'b'c'd' + a'b'c'd + a'bcd' + ab'cd' + abc'd'$	$d'(c(a'b + ab') + abc') + a'b'c'$	12	
$\mathbf{22}_{4[5]}$	4	5	192	$a'b'c'd' + a'b'c'd + a'bcd' + ab'cd' + abc'd$	$c'(a'b' + abd) + cd'(a'b + ab')$	12	
$\mathbf{23}_{4[5]}$	4	5	192	$a'b'c'd' + a'b'c'd + a'bcd' + ab'cd' + abcd$	$c(d'(a'b + ab') + abd) + a'b'c'$	12	
$\mathbf{24}_{4[5]}$	4	5	192	$a'b'c'd' + a'b'c'd + a'bcd' + ab'cd + abcd'$	$c(bd' + ab'd) + a'b'c'$	9	
$\mathbf{25}_{4[5]}$	4	5	64	$a'b'c'd' + a'b'cd + a'bc'd + a'bcd' + ab'c'd$	$a'(b(c'd + cd') + b'(c'd' + cd)) + ab'c'd$	15	
$\mathbf{26}_{4[5]}$	4	5	16	$a'b'c'd' + a'b'cd + a'bc'd + ab'c'd + abcd'$	$a(b'c'd + bcd') + a'(b'(c'd' + cd) + bc'd)$	16	
$\mathbf{27}_{4[5]}$	4	5	48	$a'b'c'd' + a'b'cd + a'bc'd + ab'cd' + abc'd'$	$ad'(b'c + bc') + a'(b'(c'd' + cd) + bc'd)$	15	

Table 2 (*continued*)

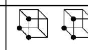
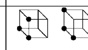
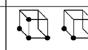
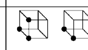
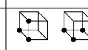
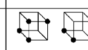
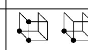
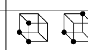
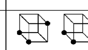
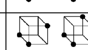
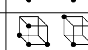
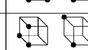
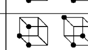
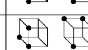
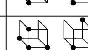
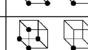
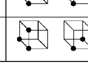
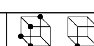
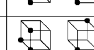
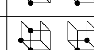
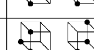
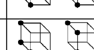
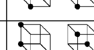
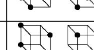
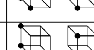
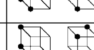
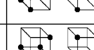
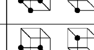
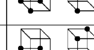
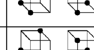
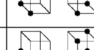
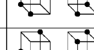
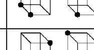
Case	*D*	*P*	*N*	*DNF*	*Minimal Formula*	*Complexity*	*Illustration*
4[6]							
$\mathbf{1}_{4[6]}$	4	6	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd$	$a'(bc' + b')$	4	
$\mathbf{2}_{4[6]}$	4	6	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bcd$	$a'(b(c'd' + cd) + b')$	7	
$\mathbf{3}_{4[6]}$	4	6	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'c'd'$	$c'd'(a'b + ab') + a'b'$	8	
$\mathbf{4}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'c'd$	$a'(b' + bc'd') + ab'c'd$	9	
$\mathbf{5}_{4[6]}$	4	6	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'cd$	$a'(b' + bc'd') + ab'cd$	9	
$\mathbf{6}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + abc'd'$	$bc'd' + a'b'$	5	
$\mathbf{7}_{4[6]}$	4	6	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + abc'd$	$bc'(a'd' + ad) + a'b'$	8	
$\mathbf{8}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + abcd$	$a'(b' + bc'd') + abcd$	9	
$\mathbf{9}_{4[6]}$	4	6	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + abc'd' + abc'd$	$abc' + a'b'$	5	
$\mathbf{10}_{4[6]}$	4	6	48	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + abc'd' + abcd$	$ab(c'd' + cd) + a'b'$	8	
$\mathbf{11}_{4[6]}$	4	6	64	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + a'bcd + ab'c'd'$	$a'(b(cd' + c'd) + b'cd)' + ab'c'd'$	13	
$\mathbf{12}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + a'bcd + ab'c'd$	$a'(b(cd' + c'd) + b'cd)' + ab'c'd$	13	
$\mathbf{13}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + a'bcd + ab'cd$	$a'(b(cd' + c'd) + b'cd)' + ab'cd$	13	
$\mathbf{14}_{4[6]}$	4	6	64	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + a'bcd + abcd$	$a'(b(cd' + c'd) + b'cd)' + abcd$	13	
$\mathbf{15}_{4[6]}$	4	6	16	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'c'd' + abcd$	$c'd'(a'b + ab') + a'b'(cd)' + abcd$	14	
$\mathbf{16}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'c'd + ab'cd'$	$b'(cd + ac'd')' + a'bc'd'$	10	
$\mathbf{17}_{4[6]}$	4	6	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'c'd + ab'cd$	$b'(ad' + a'cd)' + a'bc'd'$	10	
$\mathbf{18}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'c'd + abcd'$	$b'(a'(cd)' + adc') + bd'(ac + a'c')$	13	
$\mathbf{19}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'c'd + abcd$	$a'(b'(cd)' + bc'd') + ad(bc + b'c')$	13	
$\mathbf{20}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'cd + abc'd$	$b'(a'(cd)' + acd) + bc'(ad + a'd')$	13	
$\mathbf{21}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'cd + abcd$	$a'(b'(cd)' + bc'd') + acd$	10	
$\mathbf{22}_{4[6]}$	4	6	32	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + a'bcd' + a'bcd$	$a'(bc'd' + b'cd)'$	7	
$\mathbf{23}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + a'bcd' + ab'cd$	$a'(cd + bc'd')' + ab'cd$	10	
$\mathbf{24}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + a'bcd' + abc'd'$	$a'(cd + bc'd')' + abc'd'$	10	
$\mathbf{25}_{4[6]}$	4	6	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + a'bcd' + abc'd$	$a'(cd + bc'd')' + abc'd$	10	
$\mathbf{26}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + a'bcd' + abcd$	$a'(cd + bc'd')' + abcd$	10	
$\mathbf{27}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + ab'cd' + abc'd'$	$b'(a'(cd)' + acd') + bc'(ad' + da')$	13	
$\mathbf{28}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + ab'cd' + abc'd$	$b'(a'(cd)' + acd') + bdc'$	10	
$\mathbf{29}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + ab'cd' + abcd$	$b'(a'(cd)' + acd') + bd(ac + a'c')$	13	
$\mathbf{30}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + ab'cd + abc'd'$	$b'(a'(cd)' + acd) + bc'(ad' + da')$	13	
$\mathbf{31}_{4[6]}$	4	6	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + ab'cd + abcd'$	$a'(b'(cd)' + bdc') + ac(bd' + db')$	13	
$\mathbf{32}_{4[6]}$	4	6	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + ab'cd + abcd$	$d(ac + a'bc') + a'b'(cd)'$	10	
$\mathbf{33}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + abcd' + abcd$	$b(ac + a'c'd) + a'b'(cd)'$	10	
$\mathbf{34}_{4[6]}$	4	6	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bcd + ab'cd + abc'd'$	$cd(a'b + ab') + a'b'(cd)' + abc'd'$	14	

Table 2 (*continued*)

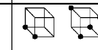
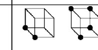
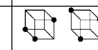
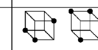
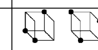
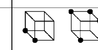
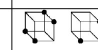
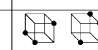
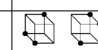
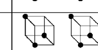
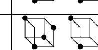
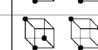
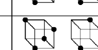
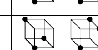
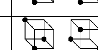
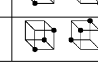
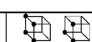
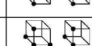
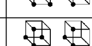
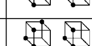
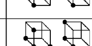
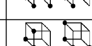
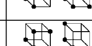
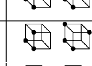
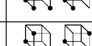
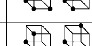
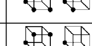
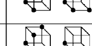
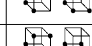
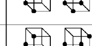
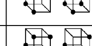
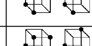
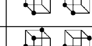
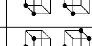
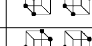
Case	*D*	*P*	*N*	*DNF*	*Minimal Formula*	*Complexity*	*Illustration*
$\mathbf{35}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bcd + ab'cd + abc'd$	$d(c(a'b + ab') + abc') + a'b'(cd)'$	13	
$\mathbf{36}_{4[6]}$	4	6	48	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bcd + ab'cd + abcd$	$cd(a'b')' + a'b'(cd)'$	8	
$\mathbf{37}_{4[6]}$	4	6	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bcd + abc'd' + abc'd$	$b(ac' + a'cd) + a'b'(cd)'$	10	
$\mathbf{38}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bcd + abc'd' + abcd$	$b(cd + ac'd') + a'b'(cd)'$	10	
$\mathbf{39}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bcd + abc'd + abcd'$	$b(d(a'c + ac') + acd') + a'b'(cd)'$	13	
$\mathbf{40}_{4[6]}$	4	6	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bcd + abc'd + abcd$	$bd(a'c')' + a'b'(cd)'$	8	
$\mathbf{41}_{4[6]}$	4	6	48	$a'b'c'd' + a'b'c'd + a'b'cd' + abc'd' + abc'd + abcd'$	$(cd)'(ab + a'b')$	6	
$\mathbf{42}_{4[6]}$	4	6	96	$a'b'c'd' + a'b'c'd + a'b'cd' + abc'd' + abc'd + abcd$	$ab(cd')' + a'b'(cd)'$	8	
$\mathbf{43}_{4[6]}$	4	6	48	$a'b'c'd' + a'b'c'd + a'b'cd' + abc'd + abcd' + abcd$	$ab(c'd')' + a'b'(cd)'$	8	
$\mathbf{44}_{4[6]}$	4	6	32	$a'b'c'd' + a'b'c'd + a'bcd' + a'bcd + ab'cd' + ab'cd$	$acb' + a'(bc + b'c')$	8	
$\mathbf{45}_{4[6]}$	4	6	96	$a'b'c'd' + a'b'c'd + a'bcd' + a'bcd + ab'cd' + abc'd'$	$ad'(b'c + bc') + a'(bc + b'c')$	11	
$\mathbf{46}_{4[6]}$	4	6	96	$a'b'c'd' + a'b'c'd + a'bcd' + a'bcd + ab'cd' + abc'd$	$a(b'cd' + bc'd) + a'(bc + b'c')$	12	
$\mathbf{47}_{4[6]}$	4	6	64	$a'b'c'd' + a'b'c'd + a'bcd' + ab'cd' + abc'd' + abcd$	$d'(c(a'b + ab') + abc') + a'b'c' + abcd$	16	
$\mathbf{48}_{4[6]}$	4	6	96	$a'b'c'd' + a'b'c'd + a'bcd' + ab'cd' + abc'd + abcd$	$cd'(a'b + ab') + abd + a'b'c'$	12	
$\mathbf{49}_{4[6]}$	4	6	48	$a'b'c'd' + a'b'cd + a'bc'd + a'bcd' + ab'c'd + ab'cd'$	$ab'(c'd + cd') + a'(b(c'd + cd') + b'(c'd' + cd))$	17	
$\mathbf{50}_{4[6]}$	4	6	8	$a'b'c'd' + a'b'cd + a'bc'd + ab'cd' + abc'd' + abcd$	$a(d'(b'c + bc') + bcd) + a'(b'(c'd' + cd) + bc'd)$	18	
4[7] $\mathbf{1}_{4[7]}$	4	7	64	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + a'bcd'$	$a'(bcd)'$	4	
$\mathbf{2}_{4[7]}$	4	7	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + ab'c'd'$	$a'(bc' + b') + ab'c'd'$	8	
$\mathbf{3}_{4[7]}$	4	7	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + ab'cd'$	$a'(bc' + b') + ab'cd'$	8	
$\mathbf{4}_{4[7]}$	4	7	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + abcd'$	$a'(bc' + b') + abcd'$	8	
$\mathbf{5}_{4[7]}$	4	7	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bcd + ab'c'd'$	$a'(b(c'd' + cd) + b') + ab'c'd'$	11	
$\mathbf{6}_{4[7]}$	4	7	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bcd + ab'c'd$	$a'(b(c'd' + cd) + b') + ab'c'd$	11	
$\mathbf{7}_{4[7]}$	4	7	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bcd + abc'd'$	$b(c'd' + a'cd) + a'b'$	8	
$\mathbf{8}_{4[7]}$	4	7	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bcd + abc'd$	$a'(b(c'd' + cd) + b') + abc'd$	11	
$\mathbf{9}_{4[7]}$	4	7	48	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'c'd' + abc'd'$	$c'd'(a'b')' + a'b'$	6	
$\mathbf{10}_{4[7]}$	4	7	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'c'd' + abc'd$	$c'(d'(a'b + ab') + abd) + a'b'$	11	
$\mathbf{11}_{4[7]}$	4	7	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'c'd' + abcd$	$c'd'(a'b + ab') + a'b' + abcd$	12	
$\mathbf{12}_{4[7]}$	4	7	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'c'd + abc'd'$	$c'(bd' + ab'd) + a'b'$	8	
$\mathbf{13}_{4[7]}$	4	7	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'c'd + abcd'$	$b'(a' + ac'd) + bd'(a'c' + ac)$	11	
$\mathbf{14}_{4[7]}$	4	7	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'cd + abc'd'$	$b'(a' + acd) + bc'd'$	8	
$\mathbf{15}_{4[7]}$	4	7	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'cd + abc'd$	$b'(a' + acd) + bc'(a'd' + ad)$	11	
$\mathbf{16}_{4[7]}$	4	7	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + abc'd' + abc'd$	$bc'(a'd)' + a'b'$	6	
$\mathbf{17}_{4[7]}$	4	7	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + abc'd' + abcd$	$b(c'd' + acd) + a'b'$	8	
$\mathbf{18}_{4[7]}$	4	7	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + abc'd + abcd'$	$b(c'(a'd' + ad) + acd') + a'b'$	11	
$\mathbf{19}_{4[7]}$	4	7	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + abc'd + abcd$	$b(ad + a'c'd') + a'b'$	8	
$\mathbf{20}_{4[7]}$	4	7	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + abc'd' + abc'd + abcd'$	$ab(cd)' + a'b'$	6	

Table 2 (*continued*)

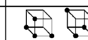
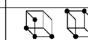
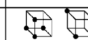
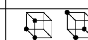
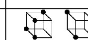
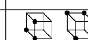
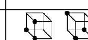
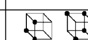
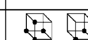
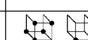
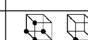
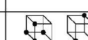
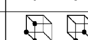
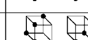
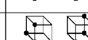
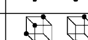
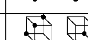
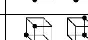
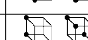
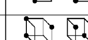
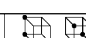
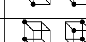
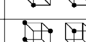
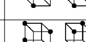
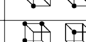
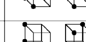
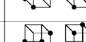
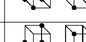
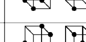
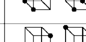
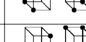
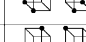
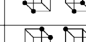
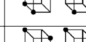
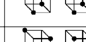
Case	*D*	*P*	*N*	*DNF*	*Minimal Formula*	*Compl.*	*Illustration*
$\mathbf{21}_{4[7]}$	4	7	96	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+a'bcd+ab'c'd'+ab'cd$	$ab'(c'd'+cd)+a'(b(cd'+c'd)+b'cd)'$	15	
$\mathbf{22}_{4[7]}$	4	7	64	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+a'bcd+ab'c'd'+abcd$	$a(b'c'd'+bcd)+a'(b(cd'+c'd)+b'cd)'$	16	
$\mathbf{23}_{4[7]}$	4	7	192	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+a'bcd+ab'c'd+ab'cd'$	$ab'(c'd+cd')+a'(b(cd'+c'd)+b'cd)'$	15	
$\mathbf{24}_{4[7]}$	4	7	384	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+a'bcd+ab'c'd+ab'cd$	$adb'+a'(b(cd'+c'd)+b'cd)'$	12	
$\mathbf{25}_{4[7]}$	4	7	192	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+a'bcd+ab'c'd+abcd'$	$a(b'c'd+bcd')+a'(b(cd'+c'd)+b'cd)'$	16	
$\mathbf{26}_{4[7]}$	4	7	192	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+a'bcd+ab'c'd+abcd$	$ad(b'c'+bc)+a'(b(cd'+c'd)+b'cd)'$	15	
$\mathbf{27}_{4[7]}$	4	7	192	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+a'bcd+ab'cd+abc'd$	$ad(b'c+bc')+a'(b(cd'+c'd)+b'cd)'$	15	
$\mathbf{28}_{4[7]}$	4	7	192	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+a'bcd+ab'cd+abcd$	$acd+a'(b(cd'+c'd)+b'cd)'$	12	
$\mathbf{29}_{4[7]}$	4	7	192	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+ab'c'd+ab'cd'+ab'cd$	$b'(ac'd'+a'cd)'+a'bc'd'$	11	
$\mathbf{30}_{4[7]}$	4	7	64	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+ab'c'd+ab'cd'+abc'd'$	$bc'd'+b'(cd+ac'd')'$	9	
$\mathbf{31}_{4[7]}$	4	7	384	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+ab'c'd+ab'cd'+abc'd$	$bc'(a'd'+ad)+b'(cd+ac'd')'$	12	
$\mathbf{32}_{4[7]}$	4	7	192	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+ab'c'd+ab'cd'+abcd$	$b(a'c'd'+acd)+b'(cd+ac'd')'$	13	
$\mathbf{33}_{4[7]}$	4	7	96	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+ab'c'd+ab'cd+abc'd$	$bc'(a'd'+ad)+b'(ad'+a'cd)'$	12	
$\mathbf{34}_{4[7]}$	4	7	384	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+ab'c'd+ab'cd+abcd'$	$bd'(a'c'+ac)+b'(ad'+a'cd)'$	12	
$\mathbf{35}_{4[7]}$	4	7	384	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+ab'c'd+ab'cd+abcd$	$b(a'c'd'+acd)+b'(ad'+a'cd)'$	13	
$\mathbf{36}_{4[7]}$	4	7	192	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+ab'c'd+abcd'+abcd$	$b'(a'(cd)'+adc')+b(ac+a'c'd')$	13	
$\mathbf{37}_{4[7]}$	4	7	64	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+ab'cd+abc'd+abcd'$	$b'(a'(cd)'+acd)+b(c'(ad+a'd')+acd')$	16	
$\mathbf{38}_{4[7]}$	4	7	192	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd'+ab'cd+abc'd+abcd$	$a'(b'(cd)'+bc'd')+ad(b'c')'$	11	
$\mathbf{39}_{4[7]}$	4	7	64	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd+a'bcd'+a'bcd+ab'cd$	$a'(bc'd'+b'cd)'+ab'cd$	11	
$\mathbf{40}_{4[7]}$	4	7	192	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd+a'bcd'+ab'cd+abc'd'$	$a(b'cd+bc'd')+a'(cd+bc'd')'$	13	
$\mathbf{41}_{4[7]}$	4	7	384	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd+a'bcd'+ab'cd+abc'd$	$ad(b'c+bc')+a'(cd+bc'd')'$	12	
$\mathbf{42}_{4[7]}$	4	7	192	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd+a'bcd'+ab'cd+abcd$	$acd+a'(cd+bc'd')'$	9	
$\mathbf{43}_{4[7]}$	4	7	192	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd+a'bcd'+abc'd'+abc'd$	$abc'+a'(cd+bc'd')'$	9	
$\mathbf{44}_{4[7]}$	4	7	192	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd+a'bcd'+abc'd'+abcd$	$ab(c'd'+cd)+a'(cd+bc'd')'$	12	
$\mathbf{45}_{4[7]}$	4	7	192	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd+a'bcd'+abc'd+abcd'$	$ab(c'd+cd')+a'(cd+bc'd')'$	12	
$\mathbf{46}_{4[7]}$	4	7	384	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd+a'bcd'+abc'd+abcd$	$abd+a'(cd+bc'd')'$	9	
$\mathbf{47}_{4[7]}$	4	7	192	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd+ab'cd'+abc'd'+abcd$	$b'(a'(cd)'+acd')+b(c'(ad'+da')+acd)$	16	
$\mathbf{48}_{4[7]}$	4	7	192	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd+ab'cd'+abc'd+abcd'$	$acd'+bdc'+a'b'(cd)'$	10	
$\mathbf{49}_{4[7]}$	4	7	384	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd+ab'cd+abc'd'+abcd'$	$b'(a'(cd)'+acd)+b(ad'+da'c')$	13	
$\mathbf{50}_{4[7]}$	4	7	384	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bc'd+ab'cd+abcd'+abcd$	$a'(b'(cd)'+bdc')+ac(b'd')'$	11	
$\mathbf{51}_{4[7]}$	4	7	192	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bcd+ab'cd+abc'd'+abc'd$	$cd(a'b+ab')+abc'+a'b'(cd)'$	13	
$\mathbf{52}_{4[7]}$	4	7	48	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bcd+ab'cd+abc'd'+abcd$	$cd(a'b')'+a'b'(cd)'+abc'd'$	12	
$\mathbf{53}_{4[7]}$	4	7	96	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bcd+ab'cd+abc'd+abcd'$	$d(c(a'b+ab')+abc')+a'b'(cd)'+abcd'$	17	
$\mathbf{54}_{4[7]}$	4	7	96	$a'b'c'd'+a'b'c'd+a'b'cd'+a'bcd+abc'd'+abc'd+abcd'$	$(cd)'(ab+a'b')+a'bcd$	10	
$\mathbf{55}_{4[7]}$	4	7	64	$a'b'c'd'+a'b'c'd+a'bcd'+a'bcd+ab'cd'+ab'cd+abc'd'$	$a(cb'+bc'd')+a'(bc+b'c')$	11	
$\mathbf{56}_{4[7]}$	4	7	16	$a'b'c'd'+a'b'cd+a'bc'd+a'bcd'+ab'c'd+ab'cd'+abc'd'$	$a(b'(c'd+cd')+bc'd')+a'(b(c'd+cd')+b'(c'd'+cd))$	20	

Table 2 (*continued*)

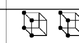
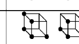
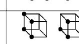
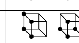
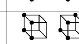
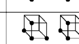
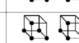
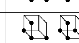
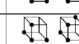
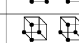
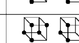
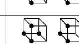
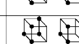
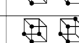
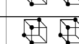
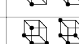
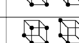
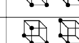
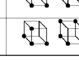
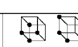
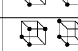
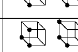
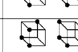
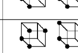
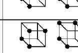
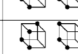
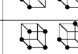
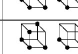
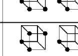
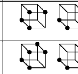
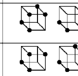
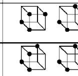
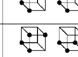
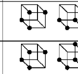
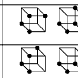
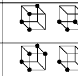
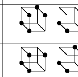
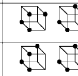
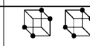
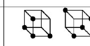
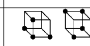
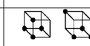
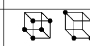
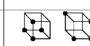
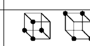
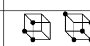
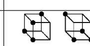
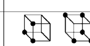
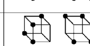
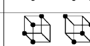
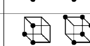
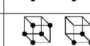
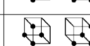
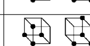
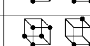
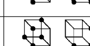
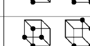
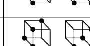
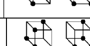
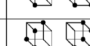
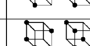
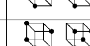
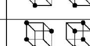
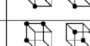
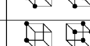
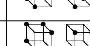
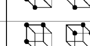
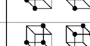
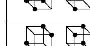
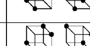
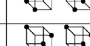
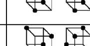
Case	*D*	*P*	*N*	*DNF*	*Minimal Formula*	*Compl.*	*Illustration*
4[8] $\mathbf{1}_{4[8]}$	4	8	8	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + a'bcd' + a'bcd$	a'	1	
$\mathbf{2}_{4[8]}$	4	8	64	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + a'bcd' + ab'c'd'$	$a'(bcd)' + ab'c'd'$	8	
$\mathbf{3}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + a'bcd' + ab'c'd$	$a'(bcd)' + ab'c'd$	8	
$\mathbf{4}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + a'bcd' + ab'cd$	$a'(bcd)' + ab'cd$	8	
$\mathbf{5}_{4[8]}$	4	8	64	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + a'bcd' + abcd$	$a'(bcd)' + abcd$	8	
$\mathbf{6}_{4[8]}$	4	8	32	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + ab'c'd' + ab'c'd$	$c'(ab' + ba') + a'b'$	7	
$\mathbf{7}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + ab'c'd' + ab'cd'$	$b'(ad' + a') + ba'c'$	7	
$\mathbf{8}_{4[8]}$	4	8	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + ab'c'd' + ab'cd$	$b'(a(c'd' + cd) + a') + ba'c'$	10	
$\mathbf{9}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + ab'c'd' + abcd'$	$a'(bc' + b') + ad'(b'c' + bc)$	10	
$\mathbf{10}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + ab'c'd' + abcd$	$a(b'c'd' + bcd) + a'(bc' + b')$	11	
$\mathbf{11}_{4[8]}$	4	8	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + ab'cd' + ab'cd$	$b'(ac + a') + ba'c'$	7	
$\mathbf{12}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + ab'cd' + abc'd'$	$b'(a' + acd') + bc'(ad)'$	9	
$\mathbf{13}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + ab'cd' + abc'd$	$b'(a' + acd') + bc'(ad')'$	9	
$\mathbf{14}_{4[8]}$	4	8	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + ab'cd' + abcd'$	$a'(bc' + b') + acd'$	7	
$\mathbf{15}_{4[8]}$	4	8	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + ab'cd' + abcd$	$a'(bc' + b') + ac(b'd' + bd)$	10	
$\mathbf{16}_{4[8]}$	4	8	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bc'd + abcd' + abcd$	$b(ac + a'c') + a'b'$	7	
$\mathbf{17}_{4[8]}$	4	8	48	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bcd + ab'c'd' + ab'cd$	$ab'(c'd' + cd) + a'(b(c'd' + cd) + b')$	13	
$\mathbf{18}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bcd + ab'c'd' + abc'd'$	$a'(b' + bcd) + c'd'(a'b')'$	9	
$\mathbf{19}_{4[8]}$	4	8	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bcd + ab'c'd' + abc'd$	$ac'(b'd' + bd) + a'(b(c'd' + cd) + b')$	13	
$\mathbf{20}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bcd + ab'c'd' + abcd$	$c'd'(a'b + ab') + bcd + a'b'$	11	
$\mathbf{21}_{4[8]}$	4	8	48	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bcd + ab'c'd + ab'cd'$	$ab'(c'd + cd') + a'(b(c'd' + cd) + b')$	13	
$\mathbf{22}_{4[8]}$	4	8	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bcd + ab'c'd + abc'd'$	$b'(a' + ac'd) + b(c'd' + a'cd)$	11	
$\mathbf{23}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bcd + ab'c'd + abc'd$	$a'(b(c'd' + cd) + b') + adc'$	10	
$\mathbf{24}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bcd + ab'c'd + abcd'$	$a(b'c'd + bcd') + a'(b(c'd' + cd) + b')$	14	
$\mathbf{25}_{4[8]}$	4	8	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bcd + abc'd' + abc'd$	$a'(b' + bcd) + bc'(a'd)'$	9	
$\mathbf{26}_{4[8]}$	4	8	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bcd + abc'd' + abcd$	$b(cd + c'd') + a'b'$	7	
$\mathbf{27}_{4[8]}$	4	8	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + a'bcd + abc'd + abcd'$	$b(a(c'd + cd') + a'(c'd' + cd)) + a'b'$	13	
$\mathbf{28}_{4[8]}$	4	8	48	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'c'd' + abc'd' + abcd$	$c'd'(a'b')' + a'b' + abcd$	10	
$\mathbf{29}_{4[8]}$	4	8	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'c'd' + abc'd + abcd'$	$c'(d'(a'b + ab') + abd) + a'b' + abcd'$	15	
$\mathbf{30}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'c'd' + abc'd + abcd$	$c'd'(a'b + ab') + abd + a'b'$	11	
$\mathbf{31}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'c'd + abc'd' + abc'd$	$b'(a' + ac'd) + bc'(a'd)'$	9	
$\mathbf{32}_{4[8]}$	4	8	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'c'd + abc'd' + abcd'$	$b'(a' + ac'd) + bd'(a'c)'$	9	
$\mathbf{33}_{4[8]}$	4	8	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'c'd + abc'd' + abcd$	$ad(b'c' + bc) + bc'd' + a'b'$	11	
$\mathbf{34}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'c'd + abcd' + abcd$	$b'(a' + ac'd) + b(ac + a'c'd')$	11	
$\mathbf{35}_{4[8]}$	4	8	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'cd + abc'd' + abc'd$	$b'(a' + acd) + bc'(a'd)'$	9	
$\mathbf{36}_{4[8]}$	4	8	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'cd + abc'd' + abcd$	$acd + bc'd' + a'b'$	8	
$\mathbf{37}_{4[8]}$	4	8	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + ab'cd + abc'd + abcd'$	$b'(a' + acd) + b(c'(a'd' + ad) + acd')$	14	
$\mathbf{38}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + abc'd' + abc'd + abcd'$	$b(c'(a'd)' + acd') + a'b'$	9	
$\mathbf{39}_{4[8]}$	4	8	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + abc'd' + abc'd + abcd$	$b(c'(a'd)' + acd) + a'b'$	9	
$\mathbf{40}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + a'bc'd' + abc'd + abcd' + abcd$	$a'(b' + bc'd') + ab(c'd')'$	9	

Table 2 (*continued*)

Case	*D*	*P*	*N*	*DNF*	*Minimal Formula*	*Co.*	*Illustration*
$\mathbf{41}_{4[8]}$	4	8	12	$a'b'c'd' + a'b'c'd + a'b'cd' + a'b'cd + abc'd' + abc'd + abcd' + abcd$	$ab + a'b'$	4	
$\mathbf{42}_{4[8]}$	4	8	64	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + a'bcd + ab'c'd' + ab'cd + abc'd$	$a(b'(c'd' + cd) + bc'd) + a'(b(cd' + c'd) + b'cd)'$	18	
$\mathbf{43}_{4[8]}$	4	8	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + a'bcd + ab'c'd' + ab'cd + abcd$	$a(cd + b'c'd') + a'(b(cd' + c'd) + b'cd)'$	15	
$\mathbf{44}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + a'bcd + ab'c'd + ab'cd' + ab'cd$	$ba'(c'd' + cd) + b'(ac'd' + a'cd)'$	13	
$\mathbf{45}_{4[8]}$	4	8	64	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + a'bcd + ab'c'd + ab'cd' + abc'd'$	$a(b'(c'd + cd') + bc'd') + a'(b(cd' + c'd) + b'cd)'$	18	
$\mathbf{46}_{4[8]}$	4	8	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + a'bcd + ab'c'd + ab'cd' + abc'd$	$a(dc' + b'cd') + a'(b(cd' + c'd) + b'cd)'$	15	
$\mathbf{47}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + a'bcd + ab'c'd + ab'cd' + abcd$	$a(b'(c'd + cd') + bcd) + a'(b(cd' + c'd) + b'cd)'$	18	
$\mathbf{48}_{4[8]}$	4	8	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + a'bcd + ab'c'd + ab'cd + abc'd$	$ad(bc)' + a'(b(cd' + c'd) + b'cd)'$	13	
$\mathbf{49}_{4[8]}$	4	8	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + a'bcd + ab'c'd + ab'cd + abcd'$	$a(db' + bcd') + a'(b(cd' + c'd) + b'cd)'$	15	
$\mathbf{50}_{4[8]}$	4	8	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + a'bcd + ab'c'd + ab'cd + abcd$	$ad(bc')' + a'(b(cd' + c'd) + b'cd)'$	13	
$\mathbf{51}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + a'bcd + ab'c'd + abcd' + abcd$	$a(bc + b'c'd) + a'(b(cd' + c'd) + b'cd)'$	15	
$\mathbf{52}_{4[8]}$	4	8	64	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + a'bcd + ab'cd + abc'd + abcd'$	$a(d(b'c + bc') + bcd') + a'(b(cd' + c'd) + b'cd)'$	18	
$\mathbf{53}_{4[8]}$	4	8	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + a'bcd + ab'cd + abc'd + abcd$	$ad(b'c')' + a'(b(cd' + c'd) + b'cd)'$	13	
$\mathbf{54}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'c'd + ab'cd' + ab'cd + abc'd'$	$bc'd' + b'(ac'd' + a'cd)'$	10	
$\mathbf{55}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'c'd + ab'cd' + ab'cd + abc'd$	$bc'(a'd' + ad) + b'(ac'd' + a'cd)'$	13	
$\mathbf{56}_{4[8]}$	4	8	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'c'd + ab'cd' + ab'cd + abcd$	$b(a'c'd' + acd) + b'(ac'd' + a'cd)'$	14	
$\mathbf{57}_{4[8]}$	4	8	64	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'c'd + ab'cd' + abc'd' + abcd$	$b(c'd' + acd) + b'(cd + ac'd')'$	12	
$\mathbf{58}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'c'd + ab'cd' + abc'd + abcd'$	$b(c'(a'd' + ad) + acd') + b'(cd + ac'd')'$	15	
$\mathbf{59}_{4[8]}$	4	8	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'c'd + ab'cd' + abc'd + abcd$	$b(ad + a'c'd') + b'(cd + ac'd')'$	12	
$\mathbf{60}_{4[8]}$	4	8	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'c'd + ab'cd + abc'd + abcd'$	$b(c'(a'd' + ad) + acd') + b'(ad' + a'cd)'$	15	
$\mathbf{61}_{4[8]}$	4	8	384	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'c'd + ab'cd + abcd' + abcd$	$b(ac + a'c'd') + b'(ad' + a'cd)'$	12	
$\mathbf{62}_{4[8]}$	4	8	32	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd' + ab'cd + abc'd + abcd' + abcd$	$bd'(a'c' + ac) + ad(b'c')' + a'b'(cd)'$	14	
$\mathbf{63}_{4[8]}$	4	8	32	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + a'bcd' + a'bcd + ab'cd + abc'd'$	$a(b'cd + bc'd') + a'(bc'd' + b'cd)'$	14	
$\mathbf{64}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + a'bcd' + ab'cd + abc'd' + abc'd$	$a(bc' + b'cd) + a'(cd + bc'd')'$	12	
$\mathbf{65}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + a'bcd' + ab'cd + abc'd' + abcd$	$a(cd + bc'd') + a'(cd + bc'd')'$	12	
$\mathbf{66}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + a'bcd' + ab'cd + abc'd + abcd'$	$a(d(b'c + bc') + bcd') + a'(cd + bc'd')'$	15	
$\mathbf{67}_{4[8]}$	4	8	192	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + a'bcd' + ab'cd + abc'd + abcd$	$ad(b'c')' + a'(cd + bc'd')'$	10	
$\mathbf{68}_{4[8]}$	4	8	48	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + a'bcd' + abc'd' + abc'd + abcd'$	$ab(cd)' + a'(cd + bc'd')'$	10	
$\mathbf{69}_{4[8]}$	4	8	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + a'bcd' + abc'd + abcd' + abcd$	$ab(c'd')' + a'(cd + bc'd')'$	10	
$\mathbf{70}_{4[8]}$	4	8	24	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + ab'cd' + abc'd + abcd' + abcd$	$acd' + bd(a'c)' + a'b'(cd)'$	11	
$\mathbf{71}_{4[8]}$	4	8	96	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bc'd + ab'cd + abc'd' + abcd' + abcd$	$bc'(a'd + ad') + ac(b'd')' + a'b'(cd)'$	14	
$\mathbf{72}_{4[8]}$	4	8	48	$a'b'c'd' + a'b'c'd + a'b'cd' + a'bcd + ab'cd + abc'd' + abc'd + abcd'$	$(cd)'(ab + a'b') + cd(a'b + ab')$	12	
$\mathbf{73}_{4[8]}$	4	8	8	$a'b'c'd' + a'b'c'd + a'bcd' + a'bcd + ab'cd' + ab'cd + abc'd' + abc'd$	$a(bc' + cb') + a'(bc + b'c')$	10	
$\mathbf{74}_{4[8]}$	4	8	2	$a'b'c'd' + a'b'cd + a'bc'd + a'bcd' + ab'c'd + ab'cd' + abc'd' + abcd$	$a(b(c'd' + cd) + b'(c'd + cd')) + a'(b(c'd + cd') + b'(c'd' + cd))$	22	

objects are judged as members of the target concept, as well as the confidence with which the trained objects are correctly classified. Thus such studies have usually not involved comparisons with any other concept types. In this vein, Medin and Schaffer (1978) studied type $\mathbf{12}_{4[5]}$ and (separately) $\mathbf{24}_{4[6]}$. Medin, Altom, Edelson, and Freko (1982) and several later studies (see Nosofsky et al., 1994a) used $\mathbf{11}_{4[4]}$.

For $D = 2$, many studies have found that affirmation ($\mathbf{1}_{2[2]}$) is easier to learn than conjunction or disjunction ($\mathbf{1}_{2[1]}$), which is in turn easier than exclusive-or biconditional ($\mathbf{2}_{2[2]}$); see Bourne (1970), Bourne, Ekstrand, and Montgomery (1969), and Haygood and Bourne (1965) for surveys and discussion of this extensive literature. Neisser and Weene (1962) and Haygood (1963) suggested that these rank orderings might reflect differences in inherent logical complexity, but this proposal was not pursued in the subsequent literature. Shepard et al. (1961), in their study of the 3[4] family, found that the concepts had subjective difficulties in the order $\mathbf{1}_{3[4]} < \mathbf{2}_{3[4]} < [\mathbf{3}_{3[4]}, \mathbf{4}_{3[4]}, \mathbf{5}_{3[4]}] < \mathbf{6}_{3[4]}$, with $\mathbf{3}_{3[4]}$, $\mathbf{4}_{3[4]}$ and $\mathbf{5}_{3[4]}$ of approximately equal difficulty. This result has been replicated several times (e.g., Nosofsky et al., 1994b) (see Table 2).

Feldman (2000) studied the subjective difficulty of the 41 types in families 3[2], 3[3], 3[4], 4[2], 4[3], and 4[4], and found that it is generally correlates with Boolean complexity. This idea, when it had been proposed in slightly different forms by both Neisser and Weene (1962) and Haygood (1963), had not met with acceptance in the psychological literature. In retrospect this was probably due to the fact that studies at the time were limited almost exclusively to the families 2[1] and 2[2], where this pattern is not yet apparent. It is hoped that such misapprehensions might in the future be avoided due to the availability of a catalog such as that given here.

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