1972

THE NATURE AND DEVELOPMENT OF EARLY NUMBER CONCEPTS[1]

Rochel Gelman
UNIVERSITY OF PENNSYLVANIA

[1] The research reported in this paper and the preparation of the manuscript were supported by NICHHD Grant No. 04598. The author wishes to thank the staffs of the Garrettford Elementary School, Upper Darby, Pennsylvania, Trinity Nursery School, Paoli, Pennsylvania, and the Melrose Hall School for Boys and Girls and YWCA Chestnut House of Philadelphia, Pennsylvania; Marsha Freifelder and Denise

PP1 (35).max

I. Introduction

The purpose of this paper is twofold: (1) To provide a conceptual framework within which to study the early development of concepts of quantity; and (2) to demonstrate how this framework can be used to gain insight into the young child's conception of number. Particular emphasis is placed on the problem of studying number concepts in preschool children. Although much of the paper will deal with the development of the concept of number, it is thought that the general approach can serve as a working model for an understanding of the acquisition of other quantity concepts.

A. Basic Concepts: Estimator and Operator

We begin by drawing a distinction between the process of extracting an *estimate* of a given quantity and the process by which one judges the consequences of transforming a quantity. The cognitive processes by which people determine some quantity, such as the numerosity of a set of objects, are termed *estimators*. The cognitive processes by which people determine the consequences of transforming a quantity in various ways are termed *operators*. Thus, the distinction is between processes that are used to determine how many items there are in an array or the relative numerosity of two arrays as against the processes used to determine whether transformations performed on a set affect its numerosity.

There are several reasons for distinguishing between operators and estimators. First, estimators may be thought to involve a "lower" level of processing than operators. Estimators are closely tied to perceptual input, operators are not. Operators can, at least in principle, specify the outcome of applying a particular operation to a given amount whether or not the amount is actually present. For example, adults assume that pouring a quantity of water from one container to another will not change the quantity, whether or not they are actually witnessing such a transformation of some quantity of water. In contrast, they cannot estimate accurately some quantity of water unless water is actually present to be estimated.

Second, operators may be thought to be "more cognitive" than estimators. Operators provide integrative links between successive estimates. Thus, for example, a perception that five rabbits went into the hat, in conjunction with the perception that only two rabbits came out of the hat, provides surprise through the mediation of a cognitive operator which specifies that placing arrays in hats and taking them out again are not transformations that alter number. In this case, two successive estimates of number are integrated by means of a cognitive operator.

A third reason for regarding the distinction between estimators and operators as important is that operators are more central to a mature conception of number. Consider the category of transformations which involve displacements or rearrangements. Adults, who treat number operationally, do not believe that the numerosity of a set is altered by such transformations, no matter how great the perceptual changes produced by them. And, as Piaget (1952) has shown, young children give evidence of believing that such transformations alter the quantity. In Piaget's view it is the presence of operators in the adult that enables him to judge numbers as invariant under such transformations. Similarly, Piaget argues that young children fail to judge number as invariant under such transformations because they lack the appropriate operators. And because they lack the appropriate operators they are said to have not yet developed a concept of number. The centrality of operators in Piaget's theory can be seen from the fact that children are said not to have a concept of number until they pass the operator task, i.e., conservation. Yet most children can accurately *estimate* smaller numbers by counting well before they pass the conservation test. Piaget's assumptions are reflected in several scalogram analyses of the number concept which assign a higher developmental ranking to operational tasks than estimation tasks (Siegel, 1971; Wohlwill, 1960).

Consideration of experiments designed to investigate the development of operators highlights a fourth reason for distinguishing between operators and estimators. It appears that the way in which an individual estimates at any given time influences the way he behaves in such experiments, in particular, whether he uses operators. This observation leads to the thesis that the use of the two types of processes is related in a complex fashion, and that a consideration of how a particular estimation strategy process influences the use of operators would provide guidelines for studying the development of number concepts in young children.

To illustrate that the use of one estimation strategy as opposed to another is related to the use of operators, we start by considering a hunter who sees a flock of ducks spread out in the sky. When he first sees the flock, he estimates the number of ducks at 150. After watching the same

Weinberg, who collected the data; and Justin Aronfreed, C. R. Gallistel, Ellen Markman, Dan Osherson, and Burton Rosner who carefully read earlier versions of this manuscript. I am particularly grateful to my husband, C. R. Gallistel, who took the time to help elucidate some of the concepts in the paper and translate foreign manuscripts.

PP1 (35).max

flock land close together on the small pond nearby, he changes the estimate to 75. In this particular situation, the ducks simply rearranged themselves. No ducks flew away. Yet the effect of the rearrangement led the hunter to say there were fewer ducks. Why? One explanation is that he thought the number actually changed. Yet questioning the hunter would probably show he did not think this. Instead, he would say that his original estimate was a guess based on the extent of the array and that the landing of the ducks led him to alter his initial guess. Further, he would argue that this was a reasonable thing to do since he had no reason to be confident about his initial estimate. However, had the hunter reason to be confident about his initial estimate, he would not have changed it. Thus, for example, if he could have counted the ducks or been told by someone what the actual number of ducks was, he would not have suggested that there were fewer ducks after they rearranged themselves.

The example serves to illustrate that whether operators are revealed in behavioral reactions to transformations depends on the type of estimator invoked. When an estimate is based on a property that is not invariant across transformations of a set, the individual may treat each distribution of the set as an independent estimation trial. When the estimate depends upon numerosity *per se* then rearrangements may be ignored or deemed irrelevant. In this case, the use of invariance operators can be inferred. Thus, a consideration of estimation processes is central to our interpretation of the behavior of an observer who witnesses transformations of an array. Failure to consider these may lead one to conclude that the hunter lacked number invariance operators or that he treated rearrangement as a transformation that is relevant to number. These considerations point to the possibility that, to a large extent, the *apparent* absence or presence of number invariance operators may depend upon the absence or presence of the ability to estimate number with precision and confidence. If one does not estimate on the basis of number *per se*, then one cannot utilize the operator part of his number scheme. Rather, one may treat the successive presentations of different arrangements of the given quantity as separate occasions for estimating.

This argument suggests several hypotheses with respect to the development of number concepts. First, the quality of estimation strategies may determine whether a child reveals a competence for treating number operationally. Second, as processes for estimating number with precision develop, so may the use of operators. Related to both of these hypotheses is the possibility that a study of processes which contribute to a child's confidence in his estimates may aid our understanding of developing quantity concepts.

In order to assess these hypotheses it is necessary to study the development of estimators as well as operators. In general, these two processes have not been considered together in investigations of developing number concepts. The interest in the development of number concepts can be traced back to the end of the nineteenth century. Initially, psychologists tended to investigate processes that are classified as estimators in this paper. Thus, much of the early work focused on how children count, perceive, and discriminate numbers. Although there was some discussion of operations, particularly by McLellan and Dewey (1896), Piaget's treatise on number ushered in a concerted effort to investigate operations. The earlier work did not place questions of what children knew about addition and subtraction into the framework of assessing development of operators but rather into a catalogue of number facts which children of various ages knew. Thus, for example, Brownwell (1941) assessed children's knowledge of number combinations such as $(1 + 2)$ or $(5 - 3)$ and provided normative charts showing which of these number facts children knew. The studies were not designed to determine the operations which children thought did or did not change number. Piaget shifted the framework from one in which the child's responses to number were thought to indicate mastery of number facts to one in which the child's responses to number were thought to reveal the functioning of underlying cognitive operators. Thus, further studies of children's comprehension of addition and subtraction (e.g., Smedslund, 1966a, 1966b, 1966c; Wohlwill, 1960) were placed in the context of Piaget's treatment of number as a manifestation of logical thinking.

Although most of the research that is relevant to the discussion of estimators predates Piaget's work, some work in this area has continued (e.g., Beckwith & Restle, 1966; Potter & Levy, 1968). And, as indicated below, some of the current research on the pre-conserver's notion of number is relevant to the discussion of estimators (e.g., Gelman, 1969a; Mehler & Bever, 1967; Wohlwill, 1960; Zimiles, 1963, 1966). Still, there has been little tendency to consider the two problems together and to take into account possible interactions.

The body of this paper begins with a review of the literature that pertains to estimators and a consideration of methodological problems in determining the properties upon which the child's estimate is based. There then follows a report of research on estimators from our own laboratory. Then we take up the problem of confidence in estimates. Finally, there is a selected review of previous research on operators and a summary of our current studies on operators that have been designed in light of the results of our estimation studies.

PP1 (35).max

II. Estimators

A. Methodological Considerations

Estimator tasks are those tasks which were designed to determine under what conditions subjects can abstract the precise or approximate numerosity of a perceptual array. In such tasks, subjects are asked to judge, construct, or compare arrays which vary in numerosity and other attributes.

Various tasks of this kind have been used with children. For present purposes, it is necessary to decide whether the child's responses in these tasks were based on the numerosity or on other distinctive cues in the array(s). Knowing which cues determined the responses will help one determine what estimation processes the child utilized. Therefore, the question is what methodologies can or cannot be used to assess the extent to which young children respond to numerosity *per se*. An analysis of the tasks used by various investigators to study number concepts in children helps answer the question.

Some investigators use a discrimination paradigm, in which children are to respond differentially to two arrays of equal or unequal number. In some of the number discrimination studies, children are required to learn to pick the larger array (e.g., Long & Welch, 1941; Wohlwill, 1962). In others, they are asked to decide whether the arrays contain the same number or if one has more or less than the other (e.g., Rothenberg, 1969; Russell, 1936; Zimiles, 1966).

Generally, the above discrimination tasks do not indicate which attribute of the arrays was responded to by the child. Consider what interpretations can be made of "correct" responses to differences, i.e., reaching a learning criterion of choosing the array which has more (or less) objects. Figure 1 presents schematic samples of pairs of arrays that have been used in various experiments. In some cases these are placed side by side (e.g., Long & Welch, 1941; Russell, 1936); in other cases one above the other (e.g., Rothenberg, 1969; Zimiles, 1966). The actual numbers in each stimulus pair have been varied, e.g., 3 vs. 5, 6 vs. 10, 1 vs. 6, etc. Figure 1 shows that each pair of arrays always has at least one other difference redundant with the difference in numerosity. This is necessarily so when only two stimuli are used. Consider the Pairs A and B in Fig. 1 where the two arrays each have different numerosities but no differences in size, shape, or color of the objects or the pattern of dispersal. If the two arrays are of equal length (Pair B), then they must vary in density and the more dense stimulus must be the more numerous. Conversely, if density is held constant (Pair A), length varies with numerosity. In stimulus pairs C and D length and density are in conflict, respectively, with numerosity (cf. Zimiles, 1966);

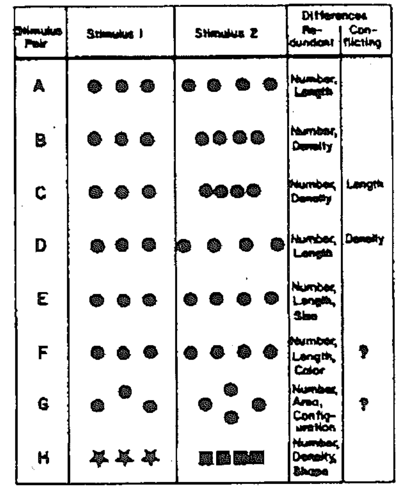

Fig. 1. Schematic illustration of representative arrays used in discrimination of number tasks where the numbers differ.

however, density and length, respectively, are redundant with numerosity. Thus, for example, if Stimulus 2 in Pair C is chosen as "more," one cannot be certain as to whether density or numerosity prevailed over length. Thus, children may score correctly without responding to number.

A related problem occurs when the number of objects in both arrays is equal. Figure 2 shows eight stimulus pairs. Each array in a pair has the same number. Inspection makes it clear that many stimulus pairs can be judged "same" on grounds other than number. Only the arrays of Pair H are alike on number alone; and this is true only if the child responds to the whole array as opposed to parts of arrays. If he does the latter, he could judge Pair H the same because they both contain a small black circle. Thus, in many cases a child could again score correctly without responding to number. Other cues are available.

The presence of redundant non-numerical cues confounds the interpretation of performance in other number tasks as well. A child required to reproduce a given set of objects for number (Brownwell, 1941; Buckingham & MacLatchy, 1930) may simply pair off each object without assessing the number (Piaget, 1952). Or a child required to identify, for example, which of three arrays contains 10 objects (Brownwell, 1941; Grant, 1938; Woody, 1931) may solve the problem by remembering that 10 means a lot and by looking for the biggest array (Brownwell, 1941). Although both

PP1 (35).max

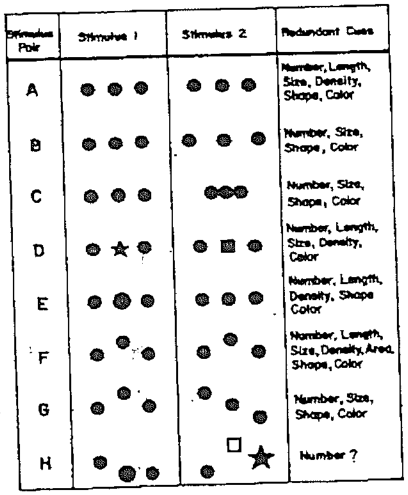

Fig. 2. Schematic illustration of representative arrays used in discrimination of number tasks where the numbers are the same.

of these strategies are "quantitative" to some degree, neither reveals the unambiguous use of number alone.

The difficulty in inferring whether a child responds to number in the preceding tasks is reduced or disappears when other response criteria are included. For example, verbal protocols may serve this end. The child might be asked why x has more, why x equals 10, or why x is the winner. In a study described later, the child's ability to respond to number was assessed with a modified discrimination task. Children aged 3–5 years were shown two arrays of toy mice differing in number (2 vs. 3) and length *or* density. The child had to determine which array was the "winner" and which the "loser." During the learning, children were asked why they identified the winner or loser as such. The majority of the children said that one had 3 or the other hand 2 or both. No child said that one was longer or more bunched up.

The use of verbal protocols is not the only way to render a response unambiguous in a number task. Baldwin and Stecher (1925) noted that children who could construct a set equal in number to that in a sample, counted extensively. Observations of this sort also serve to remove ambiguity.

Not all estimation experiments present a problem of interpretation. Many have been carried out in such a way that number was the only possible cue for consistent responding. For example, Beckmann (1924) used

a number production task in which the children took out N marbles (or other objects) from a box. Since the experimenter in this task did not first make a similar sample, the child had no set to match. Therefore, it is unlikely that the children could have succeeded in this task without responding to the numerosity *per se.* Beckmann (1924) also used a simultaneous discrimination task with repeated trials. This involved presenting different examples of two numbers over a series of trials. The configurations and materials within and between the sets varied but the numbers did not. He scored children as having mastered the task only if they answered correctly regardless of configurations or materials. Thus, consistently correct responses must have been based on number; the only cue which was the same in all arrays.

A repeated trials design has also been employed in recognition experiments in which children have to identify the number of dots in arrays with varying patterns of dots (e.g., Brownwell, 1941; Freeman, 1912). Buckingham and MacLatchy (1930), in what they called a number identification task, instructed their testers to cast a given number of objects on the table before the child and then to ask how many there were. They reasoned that doing this three times for each of 1,356 children on each of nine numbers would serve to control for the use of any non-numerical cue.

This methodological discussion highlights the criteria used in selecting for review, in the next section, studies from the literature on the child's use of number. The review concentrates on those studies which reveal with some certainty whether or not the children were responding to numerosity. The experiments had to satisfy either of the following criteria: (a) the task was designed so that a correct response had to be based on the use of number, and (b) verbalizations or some other behavior of the child made it possible to decide which cues controlled the child's response.

B. Review of the Literature

1. *Studies with Preschool Children*

Monographs by Beckmann (1924) and Descoeudres (1921) contain some of the most extensive data on the ability of 2- to 6-year-old children to estimate number accurately. Baldwin and Stecher (1925) used Descoeudres' procedures with an American sample of 3- to 5-year-olds. Unfortunately, their data are reported in a way that makes it impossible to determine how closely these studies agree. Beckmann used four different tests of the child's ability to estimate number. The easiest was the number production task described in the preceding section. Despite its simplicity, it required a child to respond to number in order to succeed. Descoeudres also tested children with a number production task. However, in her test,

PP1 (35).max

the experimenter always constructed samples against which the child could compare his own. Despite some resulting ambiguity, her results are strikingly similar to Beckmann's. It seems safe to assume that the children in Descoeudres' study were in fact responding only to number. Together, these two studies surveyed 663 children. Beckmann deliberately attempted to include children from all social strata and environments in and around the city of Göttingen. His sample included children from orphanages and day-care centers for working mothers as well as children from private kindergartens and nursery schools. Although Descoeudres does not give details of her sample, it apparently included a considerable range. The normative data from these two studies on the ability to produce a specified number accurately are shown in Table I.

The percentages in these tables represent minimum figures, not average performance. In order for a given child to be counted as capable of reproducing 2, for example, the child had to perform perfectly over several trials with a variety of materials. Table I clearly shows that children between the ages of 2 and 3 years can generally produce the number 2 with considerable reliability. They seem, however, unable to produce the number 3 reliably. At some time between the age of 3 and 4 most children become capable of producing the number 3. But it is not until they are 4½ or 5 years old that they can produce larger numbers.

The similarity in the results of these two studies serves to emphasize another theme that emerges from the Descoeudres monograph: Numerosity is a very salient feature of an array for a child, *provided the array is sufficiently small so that the young child can accurately estimate its*

TABLE I

PERCENTAGE OF CHILDREN IN THE BECKMANN AND DESCOEUDRES STUDIES ABLE TO REPRODUCE EACH NUMBER CONSISTENTLY

Sample Size			Number									
			1		2		3		4		5	
B[a]	D	Age	B[b]	D	B	D	B	D	B	D	B	D
20	5	2–0	—	40	30	40	0	0	0	—	0	—
20	19	2–6	—	79	70	74	0	16	0	0	0	0
46	21	3–0	—	100	70	100	20	19	4	4	0	4
25	30	3–6	—	100	84	97	20	67	12	13	4	0
41	36	4–0	—	100	90	97	63	78	39	25	17	11
42	31	4–6	—	100	99	100	83	87	55	61	36	32
56	27	5–0	—	100	100	100	82	96	64	81	45	33
60	15	5–6	—	100	100	100	93	100	87	73	70	14
155	14	6–0	—	100	100	—	96	100	92	100	74	93

[a] B and D stand for the authors of each study.
[b] Dash indicates data not collected.

numerosity. Descoeudres went to great lengths to avoid any mention of number, yet she obtained very much the same results as Beckmann, who required the children to produce a specified number. When the children reproduced the experimenter's arrays for Descoeudres, they did so on the basis of numerosity even though their attention was not specifically directed to this property. Thus, it appears that a young child will attend to numerosity as opposed to other cues in an array if he has the capacity to estimate the number represented.

Both Beckmann and Descoeudres demonstrate that the ability to estimate number in these and other tasks is not stimulus or task specific. Beckmann, for example, trained some children to estimate reliably the number they first failed on in the normative task. Although the children were trained with a single type of stimulus material, such as marbles, their estimating ability generalized to a wide range of other objects, including objects portrayed in pictures.

Descoeudres found that the ability to discriminate arrays consistently on the basis of number is closely related to the ability to produce numbers. Table II shows a notable drop in discrimination performance when numbers greater than 3 are represented in the arrays to be discriminated. A similar effect was found in the production tasks in Descoeudres' and Beckmann's studies (see Table I). Thus, young children can differentiate sets on the basis of number if the numerosities displayed fall within a range they can estimate.

The effect shown in Tables I and II, a sharp drop in preschooler's performance when numbers greater than 3 are used, was found throughout the Beckmann and Descoeudres studies. The generality of this effect led Descoeudres to refer to what she called the "1, 2, 3, *beaucoup*" phenomena in preschool children. She used this description to indicate that the preschool child's ability to estimate number breaks down rather abruptly at

TABLE II

PERCENTAGE OF CHILDREN WHO CONSISTENTLY DISCRIMINATED ON THE BASIS OF NUMBER IN DESCOEUDRES SAMPLE[a]

	Age in years							
Discrimination	2–6 (13)[b]	3 (17)	3–6 (15)	4 (25)	4–6 (17)	5 (13)	5–6 (8)	6 (2)
1 vs. 2	23	59	67	76	100	77	100	—[c]
2 vs. 3	23	41	67	64	100	77	100	—
3 vs. 4	0	0	13	8	5	38	75	100

[a] From Descoeudres (1921, p. 278). With permission of author's publisher.
[b] Number in brackets indicates the size of each age group.
[c] Dash indicates data not collected.

PP1 (35).max

some number between 2 and 5, most typically after 3 in children younger than 4 years of age. A typical pattern of performance for a 3½-year-old child was as follows: The child could estimate the numbers 1, 2, and 3 with reliability in a variety of tasks; he could estimate the number 4, but only imprecisely and unreliably, e.g., the child sometimes produced five or three objects instead of four. For numbers greater than four the child was grossly inaccurate and unreliable. Descoeudres introduced verbal pro- tocols and other observations to demonstrate that the children regarded the larger numbers as equal to "a lot" and therefore undifferentiated from one another.

The generalization regarding the abrupt breakdown of performance at some number between 2 and 5 was true for all tasks in Beckmann's and Descoeudres' studies. However, the percentages of children passing a task for a given number varied considerably from task to task, particularly in Beckmann's studies. Many of the failures were due to extrinsic factors. For example, performance on Beckmann's most difficult task clearly de- pended on the child's ability to focus on one array of small dots shown on a paper with 11 other dot patterns. Because of these extrinsic factors, Beckmann's four estimation tasks were not scalable. The order of task difficulty for individual children could not be predicted from the group data. Beckmann's data somewhat underestimate the preschool child's ability to estimate small numbers. Since each of his tasks was an un- ambiguous test of the child's ability to respond to number *per se*, success on any one task indicated some such ability. Beckmann, however, does not give composite data indicating the percentage of children in each age group who passed at least one of the tests.

2. *Studies with School-Aged Children*

Several studies with school-aged children contained estimation tasks which allow one to determine whether children did or did not abstract numerosity from an array. Counting was discouraged in some of these studies but not in others. The latter type is emphasized in this section.

Buckingham and MacLatchy (1930) tested 6- to 6½-year-old children who were just entering the first grade and who had no previous instruction in arithmetic. They used a variety of number knowledge tasks, one of which required the children to estimate the number of objects thrown ran- domly on a table. Sets of 5, 6, 7, 8, and 10 objects were each thrown three times. Another task with the same numbers was like Beckmann's production task and required the children to hand an $N = x$ to the experimenter. Seventy percent of the children successfully estimated 10 objects on *at least one* of the three trials. The respective percentages for the numbers 8, 7, 6, and 5, were 72, 74, 75, 81. However, the ability to estimate these numbers

reliably was considerably less. Thus, only 42, 45, 46, 52, and 63% of the children correctly estimated the numbers 10, 8, 7, 6, and 5 on all three trials. Similar results were reported for the production task; for example, 85% of the children produced the number 5 at least once, but only 64% produced it all three times.

Brownwell (1928) tested first graders on their ability to indicate how many dots were present in arrays varying in number from 3–12. A given number of dots was arranged in five different two-dimensional patterns and one linear pattern. The average percentage of correct responses for the numbers 3–12 is given in Table III. Freeman (1912) had children report the number of dots they saw in a briefly exposed array. The arrays also varied in number and pattern. Freeman reported that his younger subjects (6- to 7-year-olds) could grasp the number in an array as well as older children and adults when the number represented was 4 or less. As the number increased "beyond the scope of attention" performance fell off markedly for the younger children but not for the older children and adults. He attributed this difference to the older subjects' propensity to group the objects, even when no obvious pattern was present. The older subjects used more sophisticated estimation strategies. The 6- to 7-year-old subjects did not do as well as Brownwell's subjects of the same age, probably because Brownwell's subjects were not discouraged from counting whereas Freeman's were—a point to which we will return.

Brownwell (1941) used tasks comparable to those of Buckingham and MacLatchy and obtained similar results. Thus, for example, 54% of 631 6-year-old school entrants were able to put "tails on nine of the rabbits"

TABLE III

AVERAGE PERCENTAGE CORRECT RESPONSES GIVEN TO THE NUMBERS 3–12 BY BROWNELL'S (1928) SUBJECTS

Number	Average percentage correct[a]
3	98
4	95
5	93
6	91
7	85
8	81
9	78
10	78
11	70
12	69

[a] The computations summarized here were made by the present author and are based on the data given for two-dimensional patterns. Insufficient information was given for the linear arrays.

in a picture displaying many rabbits and 80% were able to draw five marbles on an answer sheet. When instructed to give the experimenter a particular number of objects, 85, 80, 81, 80, and 77% of the children could produce 5, 6, 7, 8, and 10 objects. This finding is almost identical to that reported by Buckingham and MacLatchy for the first trial of a comparable production task.

Together these data suggest that first- and second-grade children can abstract the numerosity of arrays of 5 with no difficulty and can do so with considerable accuracy for numbers up to 9 or 10. Their ability to estimate accurately does decline as the number becomes still larger; however, this decline is not nearly so abrupt as that described for younger children.

A review of the literature on number estimation in young children has revealed two points that will reappear in subsequent sections: (1) Young children, particularly preschoolers, have difficulty estimating accurately, i.e., abstracting the numerosity of an array when the numerosity is greater than 3–5. When the numerosity is 3 or less even 2½-year-olds can frequently abstract it. (2) Provided the numerosity of an array falls within the range that a child can estimate, the numerosity is a very salient property of the array. That is, the child will attend to this property in comparing arrays even when many other readily perceived cues are also presented by the arrays. It would seem, then, that studies of the young child's ability to use number operators should involve small numbers.

C. The Role of Counting in Estimation

A considerable amount of evidence supports the hypothesis that counting[2] is the preeminent mechanism used by young children to estimate numbers of all sizes, with the possible exception of 1 and 2. It might be expected that the ability to apprehend small numbers is mediated by a "perceptual" mechanism and that this mechanism is the initial estimating process used by young children. This perceptual mechanism might rest on some apprehension of the geometric properties of the configurations made by small arrays or some at yet unexplicated process. Direct apprehension of the numerosity of small arrays without the mediation of counting is common in adults. In the literature on adult performance, it is referred to as subitizing (Beckwith & Restle, 1966; Neisser, 1966). The ability to subitize small numbers also appears in preschool children. But, as shown in Table IV, it appears to develop *after* children have learned to estimate a number by counting.

[2] A number of investigators have distinguished between the ability to repeat a number sequence by rote and the ability to coordinate the numerals with successive items. The term counting has the latter meaning here.

TABLE IV
Percentage of Children in Beckmann's Study who Counted or Subitized when Estimating Numbers[a]

Number	Process	Age of children 4–0 (48)[b]	4–6 (26)	5–0 (27)	5–6 (20)	6–0 (24)
2	Count	25	17	11	4	2
	Subitize	75	83	89	96	98
3	Count	71	33	35	21	5
	Subitize	29	67	66	79	95
4	Count	80	80	71	52	17
	Subitize	20	20	29	48	83
5	Count	100	88	76	61	33
	Subitize	—	12	24	39	67
6	Count	100	96	90	64	41
	Subitize	—	4	10	36	59

[a] From Beckmann (1924, Table 8, p. 28).
[b] Number in brackets indicates the size of each age group.

Table IV shows the results of Beckmann's analysis of the way in which children arrived at an accurate estimate of the number of items in an array. Depending on their behavior during the task and their explanations of how they reached an answer, Beckmann categorized the children as counters or as subitizers.[3] Children classified as counters were observed to count before giving their answer and said they counted when asked how they knew an answer. Children classified as subitizers responded very quickly without giving any indication of counting and said they could see when asked how they knew the answer. Thus, for example, children said "It looks like two," "I can see it's two." It can be seen in Table IV that the younger the child, the greater the tendency to count for all numbers. Furthermore, the larger the number, the greater the tendency for all children to count. Together these results support the conclusion that children estimate a number by counting before they can subitize the same number.

The same pattern of development from counting to subitizing appears in Beckmann's production task. In this task children had to produce a designated number of marbles from a box. Beckmann reported that the children used three strategies in this task: (1) Children counted as they removed one marble at a time; (2) children looked at the marbles and then pulled out

[3] The term subitize is used as a translation of Beckmann's term Erkennen.

PP1 (35).max

TABLE V
PERCENTAGE OF CHILDREN IN BECKMANN'S PRODUCTION TASK WHO USED THE DIFFERENT STRATEGIES[a]

Strategy	Age level					
	3–0/3–6	4–0	4–6	5–0	5–6	6–0
Count	58	57	46	39	43	48
Look and take	33	25	30	30	30	32
Group	8	18	24	31	27	20

[a] From Beckmann (1921, Table 6, p. 25).

the appropriate number; (3) children constructed the sum taking two groups of marbles, e.g., a child asked to produce five marbles first took two marbles and then another three. The percentage of children who used each of the strategies appears in Table V. Counting was the prevalent strategy used by children of all ages. The older children used the grouping strategy more than the younger children.

Brownwell (1928) and McLaughlin (1935) reported a similar effect. As indicated above, Brownwell tested children in grades 1 through 5 for their ability to identify the number of elements in arrays of 3–10. The arrays were presented for five seconds each, allowing sufficient time to count some of the displays. Brownwell reported that the younger children almost always counted and seldom took advantage of the patterns in the display. McLaughlin indicated that 3- to 6-year-olds typically counted in order to determine the number of objects in an array, even when the numbers were small. As the number of items a child could count increased, so did the ability to estimate. Good counters made high scores.

A comparison of performance in experiments in which children might have counted with those in which they might not have, adds support to the hypothesis that young children initially estimate by counting. We have already described Brownwell's (1928) and Buckingham and MacLatchy's (1930) work on estimation. Brownwell used a dot recognition task. The comparable Buckingham and MacLatchy test involved showing children the result of a random throw of objects. In both of these studies, children were *not* prevented from counting. As indicated, Brownwell reported that the 6-year-olds did count. Douglass (1925) used three number tasks with children of different ages, including 6-year-olds. Two of his tasks were similar to Brownwell's and Buckingham and MacLatchy's. In one, children saw 1–10 blue dots arranged in a row on cards; in the other, children had to identify the number of pennies (1–10) held in an open hand. The third task involved choosing the one of ten cards displaying the number of dots which corresponded to a spoken word between one and ten. In all three tasks, stimulus exposure was brief and the children were discouraged from

PP1 (35).max

counting. Although the three studies had samples of varying ages, all included a 6-year-old group. A comparison of Douglass' 6-year-olds with equally old children in the other two studies reveals a large discrepancy. In the studies in which the children were allowed to count, approximately 70% accurately estimated nonlinear arrays of ten on at least one trial. And although children in Brownwell's study were less likely to estimate linear arrays of 9, 10, and 11 accurately, 54% did so.[4] In general, these studies give the impression that 6-year-old children can estimate arrays of ten items with reasonable accuracy. In contrast, in Douglass' study in which children were not allowed to count, only 8% of the children could estimate the numerosity of ten element arrays. Such a low score means the children did poorly on all three of Douglass' tasks, including the two which appear comparable to those used in the other studies. Although the studies differed in a variety of ways, some of the discrepancy in estimation scores may reasonably be attributed to the presence or absence of counting.

Errors in estimation may reflect errors in counting. Russell (1936) reports that some children erred because of counting mistakes when asked to judge which of two groups of blocks had the "most." Thus, for example, a child counted an 8 vs. 9 display of blocks as 8 vs. 8 and accordingly reported the displays had the same number.

Counting may not only be the first mechanism used in estimation but may also be the basic mechanism used when children begin to add and subtract. Various investigators report that the initial stages of adding and subtracting involve counting (e.g., Beckmann, 1924; Brownwell, 1941; Ilg & Ames, 1951; Reiss, 1943). Thus, for example, a child may add 8 and 3 by counting "8, 9, 10, 11" or may even count to 8 first and then proceed to 11.

To summarize, a child's ability to abstract number seems related to his ability to count. Young children tend to count even when estimating small numbers. The ability to subitize and group elements within arrays develops later.

D. AN EXPERIMENTAL INVESTIGATION OF ESTIMATORS

1. *Introduction*

The research considered above indicates whether or not a child can abstract number and provides some evidence as to how he estimates on the basis of number. It does not, however, allow one to determine the nature of estimators used by the child who does not abstract number *per se*. This question is of some theoretical interest.

[4] The manner in which data were presented for the linear arrays allows computations of a composite score only for the numbers 9–11.

One source of this interest is Piaget's writings (Piaget, 1952, 1968). These contain the hypothesis that there are four stages in the development of estimators, and that at each of these stages children use different information in the array. Piaget provides a description of the first two stages in his interpretation of Mehler and Bever's (1967) study of the ability of 2- to 6-year-old children to compare the number of objects in two rows before and after the rows are transformed. His account of the stages in the conservation of number provides a description of the last two stages. Piaget suggests that Mehler and Bever's work shows that very young children estimate on the basis of density, judging which row is more "heaped" or "crowded," while somewhat older children estimate with length cues, judging which row is longer. According to Piaget, the early use of density cues derives from a topological intuition; and the subsequent use of length cues derives from an ordinal comparison of how far each of two rows extend. This explanation is derived from research on space concepts (Laurendau & Pinard, 1970; Piaget & Inhelder, 1956), which shows that topological notions of space develop before Euclidean ones.[5]

Piaget formulates three stages in the development of number conservation. In the first stage, children center on a perceptual cue that changes with transformations, and fail to conserve; in the second stage, they begin to consider both the length and density of arrays, and sometimes conserve. However, if a perceptual change introduced by a transformation is too great, they will focus on it and fail to conserve. Finally, in the third stage, the child recognizes the compensatory relationship between the density and length of an array and can conserve number. This account suggests that in the first stage of conservation children use length or density; then they begin to use both; and finally they can abstract number. Combining the consideration of Mehler and Bever's data on very young children and Piaget's own account of the stages in conservation in 5- to 7-year-old children, we arrive at the hypothesis that there are four stages in the development of estimators. In the first, children use density; then they use length; then length and density; and finally, number alone.

A related hypothesis has been advanced by investigators who suggest that failure on a conservation task reflects lack of a set to respond to number (e.g., Bearison, 1969; Gelman, 1969a; Zimiles, 1963, 1966). According to Zimiles and Gelman young children think of number in a multidimensional fashion. The hypothesis is that there is a hierarchy of cues controlling the young child's attention to the numerosity of an array. Included

[5] However, Piaget's definition of topology appears to differ from the mathematical system of topology in which the concept of density does not appear.

in such a hierarchy are cues like length, density, arrangement features, and the numerosity of an array. With development or experience, number is assumed to become differentiated from the other cues, and thereby more salient in the hierarchy. This position implies that experiences which serve to set young children to attend to the number and not to other features of an array will increase the likelihood of their abstracting number. Another implication is that as children develop better skills for attending to number, the size of the hierarchy controlling number estimators decreases and number *per se* becomes dominant. Older children should use number more in estimating than younger children.

To evaluate these hypotheses, one needs a method that yields the likelihood of children responding on the basis of number as well as other features of the array. One possibility is a choice procedure. However, as indicated in Section II A, two-choice procedures are problematic in that they often yield ambiguous data. To circumvent the problem of interpretation presented by a two-choice paradigm, a three-choice procedure was used. In previously assessing whether children were most likely to attend to the length, orientation, or number in a row of elements, the writer (Gelman, 1969b) employed the modified method of triads used by Suchman and Trabasso (1966a, 1966b) to study color-form preferences. Children were shown a series of triadic stimulus arrays. Each triad was constructed so that every pair of arrays in a triad was alike on a value of one binary dimension (e.g., number) and different on values of two other dimensions (length and orientation). Children had to judge which two arrays in the triad were the same. Since they received no feedback, there were no right or wrong answers. It was assumed that a child's choice of a particular pair of stimuli reflected the cue to which he attended on that trial. By considering the overall frequency of number, length, and orientation responses, it was possible to determine the likelihood that each of the cues—length, number and orientation—affected a judgment.

The study reported here also involved the triad technique. The concern was to study the development of estimators and the use of length, density, or number in an estimation task. Thus, the instruction to the children was to select the two of three stimulus arrays that had the "same number." An example of triads shown to the children is given in Fig. 3. It can be seen that each pair of arrays in a triad was alike on only one value of either a number, length, or density variable. Thus, a choice of a particular pair of arrays in response to the question of which two arrays had the same number had to be based on that variable. For example, if a child chose stimulus pair A and B it would be inferred that he responded to number. If he chose stimulus pair A and C, it would be inferred that he responded to length.

PP1 (35).max