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How Young Children Reason about Small Numbers

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INTRODUCTION

There is now considerable evidence that preschool-aged children can form accurate number-based representations of set sizes of 1–4 and sometimes 5 (Beckmann, 1924; Descoudres, 1921; Gelman, 1972a; Gelman & Tucker, 1975; Lawson, Baron, & Siegel, 1974; Smither, Smiley, & Rees, 1974). The conclusion that the representation is number based rests on the fact that the child's response—be it an absolute judgement, choice, construction of a set like the standard, etc.—can be shown to be independent of the linear extent of an array, the relative density of items in an array, and the kinds, for example, heterogeneous versus homogeneous, in an array. In other words, where care is taken to control for the possibility that the child might use nonnumerical criteria, the child is still able to perform successfully, indicating thereby an ability to use a number-based criterion.

It is now clear that the young child is able to respond to the numerosity per se of the array. This is not to say that he will always base his responses on the number of items in the array and do so without difficulty. Even within the range of 1–5 items, the larger the set the less the tendency for the young child to focus on a number-based criterion (Gelman, 1972a) or assign the correct number word (Gelman & Tucker, 1975). And the extent to which the child is influenced by heterogeneity is clearly related to experimental conditions (Gelman & Tucker, 1975; Siegel, 1973). The point simply is that the young child *can* and often does form accurate number-based representations of small sets. This is to be contrasted with the fact that larger set sizes (6 or more items) are seldom represented in terms of their numerosity. A number of investigators have documented

TABLE 1
Number of Subjects out of 48 Who Gave Correct Absolute Judgment[a]

Age and exposure time	2	3	4	5	7	11	19
	Number of items in display						
3 years							
1 sec	33	28	9	8	1	0	0
5 sec	41	38	21	16	10	1	0
1 min	41	40	28	27	20	16	5
4 years							
1 sec	44	37	23	17	4	2	1
5 sec	44	41	29	21	11	3	1
1 min	45	42	37	32	19	19	7
5 years							
1 sec	47	43	33	23	9	6	1
5 sec	44	44	37	26	19	8	2
1 min	47	46	42	38	27	19	8

[a]Details of the procedure used to obtain these data are in Gelman and Tucker (1975). The data for set sizes 2–5 were reported in that paper; those for set sizes 7, 11, and 19 are reported here for the first time.

the young child's tendency to judge the larger arrays on the basis of their length (e.g. Gelman, 1972a; Piaget, 1952; Smither *et al.*, 1974). And we find a marked decrease in the young child's ability to give accurate answers when judging the absolute number of items contained in larger arrays—even when they are given one minute to answer (See Table 1).

Why the Small Number Limit?

The fact that the young child's ability to form number based representations appears to be limited to the small number range has led to the suggestion that the young child's concept of number is "intuitive" or "perceptual" (e.g., Pufall, Shaw, & Syrdal-Lasky, 1973; Gast, 1957). My understanding of this position is two-fold: First there is the view that young children represent the number of items in an array by means of a direct perceptual apprehension mechanism, sometimes referred to as subitizing (e.g., Neisser, 1967). Second, there is the idea that young children have yet to develop an ability to reason about numbers, to understand say, that there are transformations under which the numerosity of a set remains invariant.

The hypothesis that young children might subitize the numerosity of small arrays derives in part from the assumption that adults do. In a summary of evidence from experiments on adult judgments of various set sizes (e.g., Jensen, Reese, & Reese, 1950; Kaufman, Lord, Reese, & Volkmann, 1949; Saltzman & Garner, 1948), Klahr (1973) draws attention to two factors. Adult subjects respond more quickly to small set sizes than they do large set sizes ($N \geqslant 5$ or 6). Further the reaction time (RT) function is different for the two ranges. The slope of the RT function in the small range is widely assumed to be flat (Neisser, 1967). Although upon close inspection it is not flat, it is indeed shallow—on the order of 40 msec per item (Klahr, 1973). In contrast, the slope for set sizes that are larger than five is on the order of 300 msec per item (Klahr, 1973). Figure 1 presents a schematic plot of these results. Klahr and Wallace (1973), like others (e.g., Neisser, 1967) use the difference in slopes to infer the working of different processes when subjects represent arrays of small set sizes as opposed to large set sizes. Small sets are subitized, larger sets are counted. According to Klahr and Wallace, the slight increase in the time needed to respond to each successive set size in the small number range reflects the time to retrieve the verbal label from the serial list of small number words. Thus, it is implicitly assumed that the time taken to represent each of the stimulus sets within this range is constant.

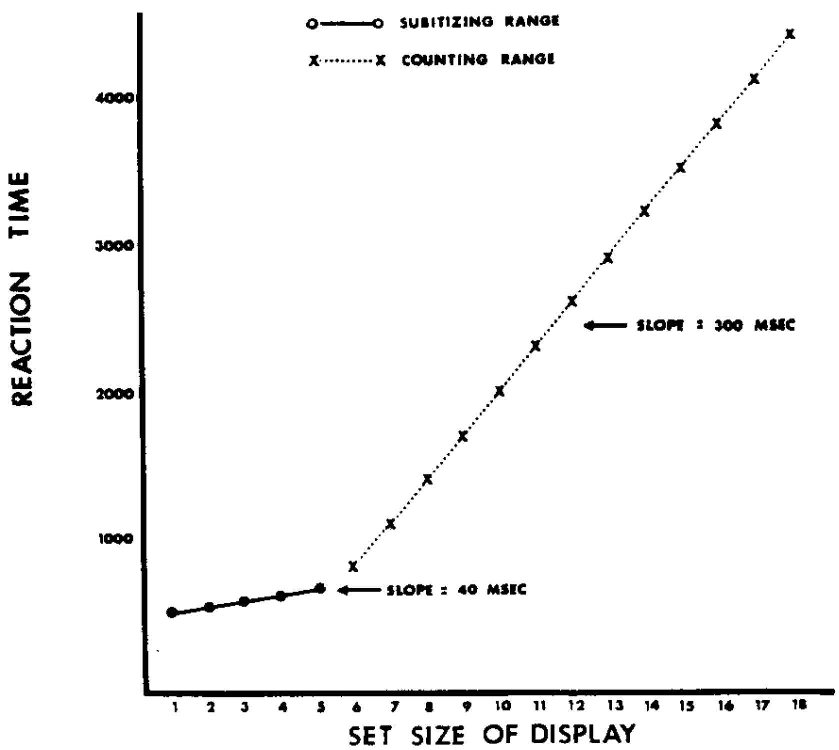

FIG. 1 Schematic presentation of the relationship between reaction time and set size. (Based on Klahr, 1973.)

Presumably then, the representation of number within this range involves a perceptual process akin to a direct apprehension one. In contrast, the authors assume that the representation process of large sets involves a counting mechanism. The counting mechanism "requires the coordination of processes that notice each object while generating the sequence of number names. When there are no more objects to be noticed, the current name is assigned to the collection of objects" (Klahr & Wallace, 1973, p. 305).

Putting together the findings that young children can represent the number of small sets but have difficulty with large sets, the inference can be made that young children subitize before they count (Klahr & Wallace, 1973). The young child's ability to deal with small numbers reflects the working of the subitizing process; their inability to deal with large numbers represents their failure to count. Presumably then, small numbers are not counted. But is this the case? The answer appears to be no.

Some years ago Beckmann (1924) advanced the hypothesis that the young child counts to represent a given small number before he shifts to relying on a subitizing-like process. Beckmann cited the fact that the younger the child, the greater his tendency to count aloud when answering questions about the number of items in an array. And I (Gelman, 1972a) noted a tendency for three- and four-year-olds to count aloud in order to determine whether unexpected variations in the properties of small sets (e.g. increases in length or number) corresponded to an actual change in number.

Counting-aloud data do not allow one to conclude that the child who counts fails to use a perceptual process like subitizing when estimating small numbers (Klahr & Wallace, 1973). They do however indicate that a young child *can* and *will* count in order to estimate. Indeed, it now seems to be the case that young children are better able to estimate small sets when tested under conditions which maximize their chances of using a counting procedure (Gelman & Tucker, 1975). Thus, for example when 3-year-olds are required to indicate how many things are in linear two-dimensional arrays of 2, 3, 4, or 5 items, the longer the exposure time of the array, the greater the accuracy scores. And, as might be expected, the greater their tendency to count aloud.

Thus, the fact that the preschooler's ability to represent numbers is limited does not appear to be due to their inability to use a counting procedure. I suspect that the assumption that the small number limit reflects a reliance on a perceptual process is what underlies Piaget's assumption that the young child's understanding of number is intuitive and not governed by a set of reasoning principles. The remainder of this chapter is given over to the presentation of my view that, even though the young child's manifested understanding of small number may be limited to the small number range, the nature of the understanding is a highly abstract one: one that contains many of the fundamental components of arithmetic reasoning. I will review the evidence on how young children reason about small numbers and outline the reasoning principles that must be available to the young child in light of these data.

A Number Reasoning Task for Young Children

Much of the evidence on how young children reason about small numbers comes from a series of "magic" experiments that I have conducted, some of which have been reported elsewhere (Gelman, 1972a, b; Gelman & Tucker, 1975) and some of which will be summarized here for the first time. Accordingly, I begin by summarizing the basic experimental paradigm. A fuller description is available in Gelman 1972b.

In all cases, the procedure involves a two-phase experiment. In the first phase, children are shown two plates containing different numbers of small plastic toys, for example, one rabbit on one plate and two rabbits on the other plate or three mice on one plate and five mice on the other plate. The experimenter designates one of these plates "the winner" while simply pointing to it—making *no reference to number.* The other plate is designated the "loser," again there is no reference to number. Phase 1 then continues as an identification game. The plates are covered with cans and the cans are shuffled. If a child appears to be keeping track of the covered winner, as in a shell game, the shuffling continues until the child appears to have lost track. When the shuffling stops, the child is asked to guess where the winner is, then lift this chosen can and verify the correctness or incorrectness of his guess. In the event that he has guessed incorrectly and recognizes this, that is, says that the uncovered plate is in fact the designated loser, he is immediately allowed to uncover the other plate. When the other plate is uncovered, he is again asked if it is the winner. The child's identification receives immediate feedback from the experimenter. If a child erroneously identified a "loser as the "winner," the experimenter says "no, that's the loser," covers the plates up again and begins reshuffing them. Whenever the child correctly identifies the winner plate as the winner, he receives verbal feedback and a prize. Then, the plates are recovered and reshuffled; the child is encouraged to help with the shuffling on half of the trials.

It should be emphasized that the feedback is based on the child's correct or incorrect identification of a plate once he has uncovered it and *not* upon his guesses. Each uncovering of a plate is counted as a trial. Thus, when a child uncovers the loser plate, correctly identifies it as such and then uncovers the winner plate, that sequence counts as two trials. The purpose of running what is basically an identification experiment in the form of a game is twofold: Firstly, the game-like nature of the task is extremely effective in engaging the young child's interest. Secondly, it builds a strong trial-to-trial expectation for one set on one plate and the other set on the other plate.

The first phase of the experiment, which we have just described, continues for at least 11 trials. On three of these trials, the child is asked to justify his identification. This serves to determine what the child takes to be the definitive properties of the sets. As reported elsewhere, the child's answers to these probes makes it clear that he almost always defines the sets in terms of their *numerosity* and he does so spontaneously. Thus, despite the fact that the experimenter makes no mention of number, the child says a plate wins (or loses) because it contains a given number of items (e.g., "It wins because it has three, 1–2–3.")

The second phase of the experiment begins when the experimenter *surreptitiously* alters some property of one or both of the expected sets. In some experiments the experimenter alters the spatial arrangement of one set, making it long and less dense or shorter and more dense. In other experiments, the experimenter alters the color or identity of one of the elements in one of the expected sets. And in still other experiments, the experimenter alters the number of elements in one or both of the expected sets, by adding or subtracting one or more elements.

From the child's standpoint, Phase 2 is just a continuation of Phase 1—until he discovers that neither plate contains a set that is identical in every way to what had been the winner set. In Phase 2 the child is also asked to identify the plates as "winner" or "loser." When the child has uncovered the altered winner plate, he is asked a series of questions: Which plate is the "winner" and why; has anything happened and if so, what; how many objects are presently on the plates, how many objects used to be on the plates; can the game proceed; does the game need fixing, and if so how it can be fixed. If the child says the game has to be fixed and suggests he needs certain items in order to do so, he is given a handful of items that include what he needs to fix the game plus several additional items.

Everything the child says is tape-recorded for later transcription. The experimenter also rates the degree of the child's surprise in Phase 2 on a 3-point rating scale (0 = no discernible surprise; 1 = some surprise; 2 = very surprised) and makes notes about any striking aspect of the child's behavior, for example, search behavior. The criteria for assigning a surprise score can be found in Gelman (1972b).

We treat the way the child responds to the surreptitious changes introduced in Phase 2 as our basic evidence for drawing conclusions about the nature of the young child's ability to reason about number. For it provides the opportunity for the child to respond to these changes in a way that involves the integration of his representation of the Phase 2 events with his established representation of the Phase 1 events. Thus for example, the child who confronts an unexpected change in number can (and does) postulate the intervention of addition or subtraction operations (e.g. "Another mice came!") to explain the discrepancy between the expected and encountered events. And reasoning about number involves the ability to integrate and relate numerical representations, one to another. Given that the child does integrate the events of Phase 1 and Phase 2, the way in which he does so can provide insight into the nature of his reasoning principles. For presumably it is the availability of reasoning principles that allows the child to decide how events x and y can be tied together.

As we have shown (e.g., Gelman 1972a; Gelman & Tucker, 1975) young children do treat the sets they encounter in Phase 2 in terms of numerical expectancies developed during Phase 1 and transformation that might have occurred between the two phases. Changes in the expected set that do *not* involve changes in number, for example, lengthening, shortening, substitution of an item of another color or identity, are often noticed. Yet they are typically classified as irrelevant with respect to number, as evidenced by the statements to the effect that the plate is still the winner-plate because it has the expected number and that the change does not matter. In contrast changes in the expected sets that *do* involve changes in number, for example, addition or subtraction are typically classified as relevant to number. For the children indicate that the altered Phase 2 set is no longer the winner-plate because it has a different number, one larger or smaller than the expected number. Moreover, they explain the change in terms of the number-relevant operations that they say must have intervened (e.g., "It's 1–2–3, three now, was two! Another one flew in. How did that happen?")

Further evidence that the event encountered in Phase 2 is integrated with the expectancy carried forward from Phase 1 comes from the children's responses to questions about fixing the game. The majority of subjects who have participated in our magic experiments are able to provide accounts as to how they (or someone else) could alter the set in a way as to make it like the original one. This is illustrated in the following protocol:

D.S. (4,7) (The subject participated in the Gelman & Tucker, 1975, experiment which involved starting with a 3-item heterogeneous–two green mice and a soldier–and a 2-item homogeneous array–two green mice. The 3-item array was designated the winner. In Phase 2 it was altered to produce a 3-item homogeneous array–three green mice.)

(Experimenter asks subject)

You take this (a mouse) *off and put on a soldier. Where's a soldier*? (Experimenter gives subject extra objects, including soldiers). *"How about two winner ones?"* (*S* places the soldier on the 2-mouse plate and says:) *"This is gonna be a winner plate too. Both have three things."*

Given that children as young as three (and sometimes two and a half) years provide such evidence, the question becomes; what is the nature of the young child's numerical reasoning? I organize the answer to this question in terms of the kinds of reasoning principles that guide the young child's ability to integrate numerical representations vis-à-vis the potential or actual effects of transformations.

THE REASONING PRINCIPLES

In this section I outline the set of arithmetic reasoning principles available to the young child. The reasoning principles that I grant the young child can be divided into three categories: (1) the relations; (2) operators; and (3) principles of "reversibility." It is my view that the observed behavior of the young child warrants the assumption that he organizes numerical comparisons in terms of the relations of equality and order; that he can classify many of the transformations that can be performed on a set as either identity preserving operations or the number changing operations of addition and subtraction; and that he has a principle of solvability which enables him to undo the effects of addition and subtraction. The latter principle and knowledge about events that can serve to cancel the effects of irrelevant transformations allow the child to relate reversible operations that can be performed on sets.

Before returning to a detailed treatment of these particular principles, it is well to indicate the domain of our inferences. The assumption that these principles are available is based on the behavioral evidence at hand. If the child's behavior is such as to warrant the granting of that principle, then the principle is granted. This is done on the assumption that the behavior could not occur were there no such principle available to the child.

Despite the fact that I require some behavioral evidence before granting a reasoning principle, I do *not* require the child to pass all potential tasks that embody that principle. As long as the child manages to perform successfully on a task in which his probability of doing so would be zero under assumption of no competence, I see no choice but to grant at least some competence. An illustrative case might be helpful.

As stated I am concerned with the question of whether preschoolers have any arithmetic reasoning principles. For the sake of argument, let us take the position that they do not. One might conclude that preschoolers lack arithmetic reasoning principles on the basis of their performance on a number conservation task, a task which they fail (Piaget, 1952). In the conservation task the child is typically shown two arrays of N items each. When the perceptual features of the array, for example, length and spacing between items are identical the young child admits to numerical equivalence. But when the perceptual properties of one array are altered, for example, one row is made longer, the young child denies that the numerosities of the arrays are still equivalent. One might conclude on the basis of such results that the young child lacks the reasoning principles that make it possible for him to treat number as invariant. But does he? In the magic experiment we demonstrate that a child of this age says a *noted* change in the length of a set of objects does not change the number therein. Further he says that addition and subtraction do change number, the former serving to increase and the latter to decrease the number of objects in an array. Add to this the fact that he never saw the transformations being performed. Rather he confronted length and number changes when he was expecting no such changes. To explain them, he himself *postulates* the intervention of the various transformations. It would seem impossible for him to do all of this without the availability of some reasoning principle(s) and supporting process(es) which organize his responses to number. He may not know how to pass the number conservation task but he obviously has some ability to treat number as invariant. What is the nature of this ability?

The Relations

In describing the quantitative relationships between the Phase 1 and Phase 2 winner-displays that children confront in the magic paradigm, the children decide that they either contain the same number and (are therefore both winners) or different numbers. If they decide the Phase 2 numerosity deviates from that encountered in Phase 1, they also indicate the direction of the deviation. Therefore I assume that when young children compare small sets x and y they recognize that the numerosities of the sets are either equal or not. If not, that is, if $x \neq y$, the children recognize that x and y satisfy an ordering relation (>) such that $x > y$ or $y > x$. The evidence for assuming the availability of these reasoning principles is as follows.

Equality. In reasoning about number, the young child recognizes numerical identity. That is, he recognizes that his representation of the numerosity of one set (the expected one) is identical to his representation of the numerosity of another set. As an immediate consequence of the recognition of identity in numerical representations, the child recognizes a numerical equivalence between sets. The evidence that children recognize an equality relation in the number domain comes primarily from those magic experiments in which the "winning" array was transformed in a fashion that was irrelevant to number. In these experiments, children unexpectedly encountered arrays (varying in set size from 2 to 5 items) that had been lengthened or shortened, or that had an item of different color or kind substituted for one of the original items. In nearly all cases, children regarded the altered array as equivalent to the original array, that is, as still the "winner." When those children who noticed the transformation were probed as to the justification for their judgment of equality, they characteristically indicated that, although some attributes of the array had changed, the number had remained the same. In other words, the reason that the altered array was the same as the original array was based on the equality of their numerosities. Thus children for example would say, "They moved out. It still wins. It's three now and it was three before."

A second line of evidence that children recognize an equivalence relation in the number domain comes from one of the studies reported by Gelman and Tucker (1975). When asked to reverse an identity change transformation, half the subjects ended up constructing *two* perceptually dissimilar arrays whose numerosities were equivalent to the numerosity of the original winner plate and said that they now had two "winners" because both had the same number as the expected "winner" plate. (See D.S.'s protocol on page 225). Notice here that the children were spontaneously constructing an equivalence based on numerosity and not simply recognizing it.

If an equality relation did not form part of the principles that guide the child's reasoning about numerosity it is difficult to understand what would lead the child to say that the numerosity of the altered set was equal to the numerosity of the original set. It would be even more difficult to understand the child's constructing a further set (out of heterogeneous items) that was equivalent in number but few, if any, other properties. Thus I assume that the child's behavior is guided by a principle which says that two numerosities may or may not satisfy an equality relation.

The assumption that an equality relation forms part of the child's reasoning principles leaves entirely open the procedure or algorithm by which the child decides whether two numerosities that he has encountered in the real world do or do not satisfy the relation. I am inclined to the position that judgments of numerical equality (and nonequality) in the magic task rest on a counting procedure.[1] For there is a noticeable tendency for the children to count when they encounter the altered array (Gelman, 1972a) and/or where they are asked to justify their judgments. As an illustration consider the answer of E.B. (3,11) who participated in a displacement condition of an experiment using a 5-mouse plate as the winner. He encountered a shortened 5-mouse plate in Phase 2 and when he did, said:

> *They crushed together.* (Is it the winner?) *Yes.* (Why?)
> *Because 1–2–3–4–5... . Were 5. Now is 5.*

Similarly children tended to confirm judgments of nonequivalence by a counting procedure as did T.P. (3,9) who said: "Can I count? 1–2–3 . . . Supposed to be 1–2–3–4 . . ." Given that they are able to make judgments of numerical nonequality, it seems reasonable to suppose they might also be able to indicate the direction of the nonequality, that is, indicate which of two arrays represents the larger set size.

Order. It appears that preschool children in fact recognize that when numerosities are not numerically equivalent, then they are numerically ordered. In other words, given two numerical representations (x and y) of two nonequivalent sets so that $x \neq y$, the child assumes that either x is more than y or y is more than x. That is the child behaves as if an ordering relation ($>$) holds between two nonequivalent numerosities.

[1] Briefly, by the counting procedure we mean the integration of (a) a one–one principle which regulates the assignment of unique tags; (b) a repeatable order principle which regulates the order of tag assignment; (c) a cardinal principle which governs the assignment of a representation to the set; and (d) a principle which serves to define what is to be counted.

Some of the evidence for this statement comes from the magic experiments in which the "winning" plate was transformed by either addition or subtraction of elements. In all of these studies children not only recognized the resulting inequality, they gave unequivocal evidence of recognizing what might be called the direction of inequality. When items had been subtracted, the children's comments and repair behavior showed that they recognized the transformed array was less than the original array. When items had been added the children understood that the transformed array was more than the original array (Gelman, 1972a, b).

Further evidence for postulating that the young child appreciates an ordering relation comes from an experiment that Merry Bullock and I are conducting this year. Again the magic paradigm is employed. In Phase 1, children are shown a 1-mouse and 2-mouse plate. Half the subjects in each age group (3- and 4-year-olds) are told the 1-mouse plate (less) wins and half are told the 2-mouse plate (more) wins. As before, there is no mention of number or quantity during Phase 1. The children spontaneously identify the winner and loser on the basis of number for example "that loses, it has 2; that wins, it has 1." In the magic phase subjects encounter a 3-mouse and a 4-mouse plate. The question is whether subjects will decide that the winner plate can be the one that honors the relationships of "more" or "less" (depending on which one they were reinforced for). The answer is yes. So far we have run 43 children between the ages of 3,0 and 4,11; all but seven decided that the "winner" plate was the one which honored the quantitative relation they were originally reinforced for. In other words, children who were reinforced for the 1-item plate in Phase 1 said the 3-mouse plate won in Phase 2; likewise those children who were initially reinforced for the 2-item array chose the 4-item array in Phase 2. There was no effect of age or reinforcement condition ($<$ or $>$) on choice behavior. Thus the children inferred that the numerical ordering relations between the winning and losing plates in Phase 1 could be generalized in Phase 2.[2]

Siegel (1974) showed that preschoolers can consistently respond to a $<$ or $>$ number relationship in a discrimination learning paradigm, indicating an ability for subjects of this age to recognize the ordering relation between simultaneously present arrays. The Bullock and Gelman experiment confirms Siegel's results. In addition, it indicates that 3- and 4-year-old children can use an ordering relation in an inferential manner. For, despite the fact that they initially judged the winning and losing arrays on the basis of their absolute set

[2] Although there was no effect of reinforcement condition or age as measured by choice behavior, one other response did show effects of these variables. Generally, when children were asked to explain their choices they were unable to do so. Still, 8 children (6 of whom were 4-year-olds) were able to provide answers that involved the use of relational terms. Given that such a term was used, it was used by the older children in the 'more' condition. Only one of these children referred to "less" in their justification; the rest referred to "more", "many", "a lot."

size they made Phase 2 choices solely on the basis of the ordering relation. When confronted with the fact that neither Phase 2 array was of the same absolute value as expected, the children chose that array which honored the same relationship as did the original winner. It should be noted that although this is an inference based upon the recognition of an ordering relation, it need not be taken as an instance of a transitive inference (Gelman & Gallistel, in preparation).

In sum, when comparing small sets young children recognize that their numerosities are either equal or not. If the sets are *not* numerically equivalent, then in the child's reasoning: If $x < y$, then the set with x items is more numerous; if $y > x$, then the set with y items is more numerous. The representations x and y appear to involve the use of a counting procedure.

The Operations

As just indicated, the magic experiments show that the child has certain numerical reasoning principles that integrate his previous experience with present experience. The recognition of numerical equivalence and order form an important part of these principles, in that they organize comparisons between present and past experiences. However, the young child's reasoning about numerosities is not limited to the drawing of comparisons. The child interprets the results of these comparisons by means of a scheme that categorizes possible real world manipulations into number-relevant and number-irrelevant ones. The possible number-relevant manipulations are subcategorized into ones that decrease and increase numerosity. The recognition that a given array is now either, more than, less than, or equivalent to the original numerosity leads the child to postulate the intervention of an operation drawn from the appropriate class in the operation classification scheme. Thus judgments of equivalence go hand in hand with reference to manipulations that do not affect numerosity and judgments of nonequivalence go along with the postulation of manipulations that do affect numerosity; judgments of nonequivalence go along with the postulation of manipulations that do affect numerosity. Since the categorization of possible manipulations in this way plays much the same role in the child's reasoning that the operations play in formal treatments of arithmetic, we refer to these categories as *operators* (cf. Gelman, 1972a).

Identity. As the magic experiments (e.g. Gelman, 1972a; Gelman & Tucker, 1975) have demonstrated, when children reason about numerosity, they recognize that there exists a large class of operations (manipulations) that can be performed on a set without altering the numerosity of the set. When called upon to explain unexpected spatial rearrangements, color changes, and item substitutions, they postulate operations having no effect on numerosity. When probed, the children will typically state that these operations do not affect numerosity.

Thus the children recognize that there exists a class of operations (which I will symbolize, *I*) and that whenever a member of this class of operations operates on a set, the numerosity of the set is not changed. It is appropriate to emphasize here that the child's behavior does not always demonstrate the existence of such a classification scheme in his number reasoning. For example, in the well-known experiments of Piaget (1952) children do not give evidence of recognizing number identity operators. However, in the magic experiments young children clearly do recognize the existence of number-identity operations, that is, operations which do not alter number.

We do not know the limits of the class of identity operators in the young child. In the adult, this class has no limit since it includes all operations except for those few that are specifically assigned to the class of number altering operations. We know that, for the young child, surreptitious transformations involving the lengthening or shortening of a linear array, changing the color, and/or identity or an item in the array are all explained in terms of identity operators. Thus, the class of identity operators is already quite extensive.

Addition and subtraction. As mentioned in the introductory remarks on operators, the young child's numerical reasoning also involves recognition of operators that *do* alter numerosity, that is, operations that are distinct from the class of identity operators. These are the operations of addition and subtraction. As summarized in Gelman (1972a; 1972b), when young children confront an unexpected increase in numerosity they postulate the intervention of addition. In other words, they state that something must have been added. Similarly in the subtraction experiments they say that something must have been removed. And that the children know that these operations alter number in a systematic way, that is, to increase or decrease it, is further demonstrated by their repair behavior. When asked how to "fix" the effect of addition they indicate that a subtraction operation is called for; likewise the effect of a subtraction operation can be repaired by adding. The following protocol demonstrates the kind of results we are referring to here.

V.B. (4,4) (Subject participated in subtraction condition of an unpublished experiment involving a 5-mouse and a 3-mouse plate in Phase 1. The 5-mouse plate was the winner and changed to a 3-mouse plate in Phase 2).

Phase 1: (Why win?) *Case there's 1–2–3–4–5.* (Why lose?) *Cause 1–2–3!*

Phase 2: (Uncovers first 3-mouse plate. Win?) *No . . . 3 mouses.* (Okay which plate wins?) She points to remaining can and lifts it. (Win?) *Wait! There's 1–2–3* (Is that the plate that wins?) *No?* (Why?) *Because it has 3. It has 3!* (What happened)? *Must have disappeared!* (What?) *The other mouses.* (Where did they disappear from?) *One was here and one was here.* She points to spaces on the nontransformed plate.–(How many now?) *1–2–3*

> (How many at beginning of game)? *There was 1 there, 1 there, 1 there, 1 there, there 1.* (How many?) *5—this one is 3 not but before it was 5*—(V. what would you need to fix the game?) *I'm not really sure because my brother is real big and he could tell* (What do you think he would need?) *Well, I don't know . . . Some things come back.* (Experimenter hands V. some objects including four mice.) V. puts all four mice on one plate. *There.. Now there's 1–2–3–4–5–6–7! No . . . I'll take these* (points to two) *off and we'll see how many.* V. Removes one and counts *1–2–3–4–5, no 1–2–3–4, Uh . . . there were 5, right?* (Experimenter says right.) *I'll put this one here* (on table), *then we 'll see how many there is now.* V. takes one off and counts *1–2–3–4–5. 5! 5.*

V. B's protocol does more than demonstrate her understanding of the role of subtraction vis-à-vis number. It also illustrates the young child's organized use of the reasoning principles. For V.B. clearly stored in memory a representation of the expected numerosity of the winner-plate. When confronted with the altered array, she obtained a representation of its numerosity. She compared the numerosity of the altered array with the stored representation of the numerosity of the winner-plate. She "decided" that the equivalence relation did not hold between these two representations of number. This decision was yoked to the conclusion that some items had been removed or "disappeared." And she revealed her knowledge about the relationship between addition and subtraction, and how they serve to undo each other.

That the children behave as if they know that addition and subtraction cancel each other leads us to postulate the availability of a reasoning principle that allows the child to "reverse" the effects of operations.

The Solvability Principle

In the magic experiments, children encountered sets whose numerosity was either more than (the addition experiments), less than (the subtraction experiments) or the same as (the displacement, color and identity change experiments) the numerosity they expected. As noted, the children reliably indicate the nature of noted discrepancies, the operations that cause a change and whether the change is relevant or irrelevant to number. I now turn attention to the fact that the children know how to eliminate the discrepancies encountered.

When asked to "fix" discrepancies, the children make cogent suggestions, ones that they are typically able to carry out. For example, children indicate that a decrease in length, "they squeezed together" can be undone by an increase in length "if you spread them out."

Or when they encounter a red mouse instead of the expected green mouse, they suggest the need for a further substitution, this time of a green mouse for the red mouse present in Phase 2. That the children know how to undo the effects of number-irrelevant transformations indicates an ability to organize various nonnumerical transformations that can be applied to events. Such knowledge in its own right does not allow us to conclude that young children know that the operations of addition and subtraction reverse each other or how such knowledge might be used. These issues require that we focus on the manner in which our subjects proceed to repair the effects of unexpected additions or subtractions.

When confronted with the discrepancy between an actual numerosity (represented as x) and an expected numerosity (y) subjects show that they know that x can be converted into the original numerosity (y) by the application of either an addition or subtraction operation. When $x < y$ they specify the need to add; when $x > y$ they talk about and engage in subtraction. In cases where addition or subtraction are called for and the difference between x and y is equal to 1 (as in all or our published experiments), the children specify not only the operation (addition or subtraction) the is appropriate but also the number of items (1) to be added or subtracted.

This latter observation, that is, subjects specify that addition (or subtraction) of 1 item is cancelled by the subtraction (or addition) of 1 item, might be taken as evidence for the position that children of this age already know that the addition or subtraction of a given number of items, x, is uniquely cancelled by the application of the reverse operation on a set size of x. This would amount to assuming that young children have a precise notion of the inverse, an assumption which I am not prepared to make. For me to take this position I would have to be able to demonstrate that young children know not only that one item has been added (or subtracted) but likewise the exact number that has been added (or subtracted) when $x > 1$. However, I have evidence to the contrary. When the difference between the expected and encountered numerosities becomes greater than 1, the children *can* indicate that some number of items greater than 1 must be added or subtracted *but* they are far from precise about the exact value of the number required. This is demonstrated in a magic study that I have already referred to here but have not published yet. Therefore I briefly summarize the details of what I call the 3 versus 5 study.

Summary of the 3 versus 5, take away 2, study. Fifty-four (30 3-year-olds and 24 4-year-olds) children were tested in Phase 1 of the magic experiment. In this phase, a 3-mouse plate, consisting of 3 green mice in a linear row, served as the "loser." A 5-mouse plate, consisting of 5 green mice in a row was identified as the "winner." As in other experiments, the difference in number was redundant to either a difference in length or density. Whether the items on each plate

TABLE 2

Summary of Phase 2 Reactions in 3 versus 5, Take Away 2 Magic Experiment

Condition and age	N	Who say they win (%)	Correct on why win or lose[a] (%)	Searchers (%)	Surprise score[b] ($\bar{X}$)	Who notice change[c] (%)	Noticers who adequately explain change[d] (%)
Subtraction							
3 years	16	12.5	75.0	68.8	1.22	93.7	80.0
4 years	16	0.0	100.0	87.5	1.44	100.0	93.7
Displacement							
3 years	8	100.0	87.5	0.0	.88	50.0	75.0
4 years	8	100.0	100.0	0.0	.88	100.0	75.0

[a]Subject is counted here only if indicates number has or has not changed from expected $N = 5$.

[b]Maximum score for subject is 2 on scale of 0 (no discernible surprise), 1 (some noticeable surprise) or 2 (very noticeable surprise).

[c]As evidenced by *any* indication of noticing, for example, surprise, explicit statement, hesitation or negative statement about the winner's status.

[d]The subject has to indicate the nature of the intervening transformations.

had the same or different distance between them was counterbalanced. Six of the 3-year-olds were dropped after Phase 1 for failing to reach the criterion of five out of six correct identifications. This left an equal number of children in each age group with the respective median ages being 3 years, 7 months and 4 years, 7 months.

Of the remaining 48 subjects, 32 were assigned to subtraction and 16 to displacement conditions and this was done so that there were equal numbers from each age group. Children in the subtraction conditions encountered a winning plate that had 3 mice, that is, two less than expected. Whether the items were removed from the ends of the original row of the second and fourth positions of the original row was counterbalanced. Children in the displacement conditions encountered a shortened or lengthened row, a factor which was also counterbalanced. All remaining details of the procedure were as outlined in the introduction. Phase 1 results compare to those we find in other studies. Thirty-four of the children made no identification errors at all. The mean number of errors for the 8 3-year-olds who erred at least once was 1.4; the mean error score for the 6 4-year-olds who erred was 1.5. As in previous experiments, there was a striking tendency for the children to sponteneously define the "winner" and "loser" in terms of their absolute number. All but six children (three in each age group) talked in terms of their numerosity. Clearly the children established an expectancy for number. How did they react to the unexpected changes they encountered in Phase 2?

In most ways, the children who participated in this experiment treated the Phase 2 events just like children who participated in other displacement versus subtraction experiments. As shown in Table 2, displacement children treated the effects of this transformation as irrelevant. They said they still won because the number of items was as expected; if they noticed the change in length they could suggest an operation that produced it. Children in the subtraction condition treated the surreptitious change in number to be a violation of their expectancy for the 5-mouse plate as winner. The altered array did not win because it only had three items; the change in number produced considerable surprise and search behavior; and it was assumed that somehow items were removed from the Phase 1 display (see V.B.'s protocol above). Thus the children behaved as we expected; they revealed an ability to make inferences about the sorts of operations that could produce the transformations they encountered.

I introduced the presentation of this 3 versus 5 take away 2, experiment by summarizing one way in which the results of this experiment differed from those in which the intervening subtraction or addition involved only one item. In the experiments where we removed or added one item, children were precise about the size of the deviation. Furthermore, they were precise about the number of items—one—that needed to be added or subtracted to "fix" the game. In the current experiment, where they expected a set size of five and encountered one with only three items, they were nowhere near as precise on these counts. They knew *some* items had been removed and generally gave evidence of knowing that it was more than one item that was missing. Thus in one of several ways 26 of 32 subtraction children talked of more than 1 missing item e.g. "*They* gone.," "*Some* came out."; "Has to be *some more*." Yet only six children could state that terms like *they, some*, and *some more* had the specific numerical reference of *two*. In other words the ability to compute, in their head, the specific number required to solve for the difference seems to be poorly developed in preschool children. Yet, that they clearly recognize that the difference can *in principle* be solved for, is shown in the way they "fix" the game.

When asked how to go about fixing the game, all but two children indicated the need to add some items. When given four mice, only four children knew to take just two of them. The rest began by taking a variable number, be it one, three or four and placing that number on a display. What followed was a sequence of counting, adding, or subtracting, counting, etc.,–much like that illustrated in V.B.'s protocol. Eventually a total of 11 of 16, 4-year-olds produced a five item array and declared it like the original. Only 4 3-year-olds met this criterion. Despite the fact that many children did not end up with a 5-mouse plate, all but three ended up with a "winning" array that had more than 3 items, arrays ranging from 4 to 7 in set size. Thus although the children knew to add items they did not necessarily know exactly how many were needed to repair the game.

Implications of the 3 versus 5 repair behavior: A principle of solvability. What does the above experiment add to our understanding of the young child's arithmetic principles? It shows that the principle that guided the young child's repair behavior in the initial magic experiments was not limited to differences of only one. Despite the fact that young children are not very good at specifying larger differences, they are able to indicate in some way that it is a difference that is greater than 1. Further they know how to begin to remove the difference. Thus in the present experiment almost all of the children in the subtraction condition knew that they should add some items. Not knowing the exact number they typically proceeded through a trial and error sequence of adding/subtracting and counting.

I hesitate to take these results as evidence for granting the children a precise concept of the inverse. Still, there is much in the behavior that warrants the postulation of some principle of reversibility, that is, some principle which leads the child to recognize that addition is what undoes the effect of subtraction and to furthermore proceed to attempt to alter the arrays in a systematic fashion. What then is the simplest principle that explains the repair behavior? I think it is a *principle of solvability*, or the "you can get there from here" principle. Put more formally, this is a principle which states the following: *given two sets* S_n *and* S_m *such that* $n < m$, *there exists a set* S_e *that when added to* S_n *will produce* S_m *and there exists a proper subset* S_d *of* S_m *that when subtracted from* S_m *will produce* S_n.

If I had evidence that the children thought the numerosity of S_e to be equal to that of S_d, I could say that they have a precise concept of the inverse. This is a question for further research. For now, the solvability principle as stated simply requires the child know to add and subtract to solve for a difference. It leaves unspecified the exact size of the difference. One might ask how it is the child arrives at the difference. We have already indicated the answer. Recall that children tend to count when working with the reasoning principles of equality and nonequality. Likewise when solving for an removing differences in numerosity children tend to employ a counting algorithm (see V.B.'s protocol). In other words, the child carries the principle of solvability into practice via algorithms that involve counting. Since the child is far from adept at these algorithms (Gelman & Gallistel, in preparation) there is many a slip between principle and practice. Yet the principle seems clearly at work.

SUMMARY

I began by focusing on the fact that the young child's ability to abstract a numerical representation of a set of objects is limited. In general, the young child seems to be able to use a number based representation for small set sizes. This observation leads some to deny the young child the ability to reason about number. Presumably there is little to reason about if the numbers are small. I accept the limited ability to abstract a numerical representation of arrays. I do not agree that this limits the child's use of small numbers to the "perceptual" or "intuitive" domain. For one, the young child can and does count small sets. Further, the evidence from magic experiments leads to the postulation of a set of arithmetic reasoning principles that young children use to reason about small numbers. These principles allow for inferences about numerical equality and order; operators that do or do not alter set size and procedures for reversing the effects of addition and subtraction. These principles may not be as advanced or sophisticated as those we attribute to older children and adults. Nevertheless they are reasoning principles. The evidence dictates some ability for the young child to reason about numbers. It seems that there is little to be gained by efforts to explain it away. Instead the effort that is called for is one that focusses on how principles like those we describe serve as the foundation for the development of more complex and extensive principles of arithmetic reasoning.

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