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What Young Children Know about Numbers

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Two sources of evidence that young children know some things about number without the benefit of school instruction are reviewed: (a) Preschoolers follow counting principles in order to represent the numerical value of a display and can reason arithmetically about numbers they represent accurately; (b) children in the early primary grades achieve an understanding that there is no largest number and that numbers never end. It is suggested that early numerical abilities, like early language abilities, are universal cognitive abilities. The notion of number readiness in terms of Piagetian theory is discussed. Consideration is given to the kind of further research required to bridge the research findings and educational practice.

That children's learning begins long before they attend school is the starting point of this discussion. Any learning a child encounters in school always has a previous history. For example, children begin to study arithmetic in school, but long beforehand they have had some experience with quantity — they have had to deal with the operations of division, addition, subtraction and size. Consequently children have their own preschool arithmetic, which only myopic psychologists could ignore. (Vygotsky, circa 1930; translated 1978)

Introduction

Note what Vygotsky took to be self-evident in the 1930's. This is that young children can know something about arithmetic without having had specific instruction in the subject. In this paper, I first will review my research on the preschool child's understanding of number, some of which is published (Gelman & Gallistel, 1978). Then I will report on research that Diane Evans and I have been doing on the development of the concept of infinity in children in Kindergarten through Grade 3. A theme that emerges from both research efforts is that there are indeed many things that young children know about arithmetic without having had the benefit of explicit school instruction. This is true for preschoolers who have not had arithmetic instruction — just as Vygotsky claimed. But it is also true for children who are in school. Although children in the early years of elementary school have lessons in arithmetic, they do not learn about infinity. Yet by the third grade they have a good understanding of the concept as it relates to the iterative process of the natural number system. Such a conclusion is a puzzlement on a variety of grounds, and I shall try to speak to some of them — especially the notions of readiness and the role of instruction in mathematics learning. But first, our research findings.

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Preschool Knowledge about Numbers

In my work on the young child's understanding of number, I make a distinction between the ability to represent the numerical value of a given set of items and the ability to reason about number. Research about the former ability involves questions as to whether a child can tell how many objects are in front of him or her; if so, what processes are used to achieve an accurate representation of the numerical value; if not, what else the child does, and so on. Research on arithmetic reasoning abilities addresses a different set of questions: Does the child know that the operations of addition increases number and subtraction decreases number? Likewise, does the child know that there is a class of transformations that are number-irrelevant? For example, does the child know that displacements of array items or the substitution of one item for another leave unaltered the numerical value of that array? One can also ask if a child knows that two different arrays which have the same number of items are equal. Likewise, one wants to determine whether the child knows that unequal values are related to each other such that one is greater than the other. And, of course, there is the question of whether the child knows that addition and subtraction are related, with one undoing the effect of the other.

The presentation of my data begins by responding first to questions about arithmetic reasoning abilities in the preschoolers. Then, I return to questions about the nature of their ability to represent the numerical value of a display.

Reasoning Abilities

I believe that preschool children know that adding and subtracting items changes the number of items in an array while the lengthening or shortening of an array does not. This is a surprising belief to those who are familiar with Piaget's (1952) findings on the development of number conservation. After all, we know that young children fail to conserve number. In the number conservation task children are shown two rows of objects, equal in length and density, placed one above the other in obvious one-to-one correspondence. Even young children typically agree that, in this case, the two rows have the same number of objects. However, young children seem not to know that the lengthening of one row is irrelevant to the numerical value of that row. For when one row is displaced, the child denies that the two rows are still numerically equivalent.

The conservation task has been criticized in a variety of ways. Perhaps the child does not understand the terms "more," "less," and "same," or perhaps the child's attention is misled by the act of lengthening a display, and so on (Gelman, 1972b). These criticisms contain the implicit assumption *that the young child does know that lengthening is a number-irrelevant transformation.* In order to demonstrate this we developed a task that we thought would less easily mislead the child — a magic task.

The basic procedure we used is as follows: There are two phases, the first is the set-up; the second, the demonstration of the results of surreptitious changes in the expected displays. In the set-up phase the child is shown two linear displays (e.g., 2 vs. 3 items) and told that one is the winner (e.g., the 3-item display). There is no mention of number; the experimenter simply points to one display and calls it the winner and points to the other display and calls it the loser. Phase I then continues as a guessing plus identification task. The displays are covered and the child guesses which one contains the winner, then looks to see if she has guessed correctly. The experimenter and child take turns covering and shuffling — usually for 10 or 11 trials. Each correct identification yields the child a prize. During Phase I, the child is occasionally asked to say why he knows he has found either the winner or the loser. The main results from Phase I are: First, it is an easy task for children even as young as 2½ years old. Second, the children tell us that this is a number game. In our example, we are often told that a 3-item display is the winner because, to quote, "It's got three, one-two-three." In other words, our set-up works. The children develop expectancies for sets of two given numerical values. This means we can continue to Phase II where children encounter the effect of a surreptitious transformation in one or both displays. The results of our first magic experiment will be presented in detail to illustrate the kind of evidence obtained.

In our first experiment (Gelman, 1972 a) we worked with 96 children, 32 in each of three age groups (3, 4, and 5 year olds). The children attended nursery schools in and around Philadelphia. The schools served a range of socioeconomic groups, still the sample tended to come more from the middle class. All children participated in a common Phase I of the experiment, where a display containing a row of three green mice was designated the winner and a display of two green mice the loser. Variations in length of displays and distance between the items were counterbalanced across children. The children were assigned to one of two main Phase II conditions, subtraction or displacement. Children in the subtraction condition encountered the effect of the surreptitious removal of one item from the winner display. Children in the displacement condition encountered the effect of the surreptitious lengthening or shortening of the winner display. Table 1 summarizes the reactions of our subjects when they saw the unexpected effect of these transformations.

Children were assigned a surprise score of 0 (no discernible surprise), 1 (minimal surprise), and 2 (moderate to extreme surprise). A hesitation, pause, or slight voice inflection or depression was rated as 1; multiple hesitations, pounding of head, screams, exclamations or any combination of these as 2. Scoring of surprise was reliable as indicated by 95% agreement between two independent raters. Children in the subtraction condition were very surprised by the change in number; those in the displacement condition showed little or no surprise to length or density changes.

Table 1 Summary of Displacement and Subtraction Children's Reactions in Phase II in First Magic Experiment[a]

	Phase II Criteria[b]				
Condition and Age	Mean Surprise Score	Noticers (N)	Searchers (N)	Noticers Who Explain Adequately (N)	Noticers Scored Reversers (N)
Subtraction:					
3 years..........	1.44	16	16	14	12
4 years..........	1.20	16	13	16	12
5 years..........	1.75	16	13	15	13
Displacement:					
3 years..........	0.25	10	0	8	5
4 years..........	0.50	8	0	8	5
5 years..........	0.50	7	0	5	5

[a]From Gelman (1972a, p. 82); used here with permission.

[b]The criteria were used as follows: Surprise Score — see text; Noticer — scored if there was any evidence of treating Phase II differently than Phase I; Searcher — scored if child looked on floor, around room and the like *or* asked a search question; Explain Adequately — subtraction children had to say something that revealed a belief that subtraction had occurred; displacement children had to show or indicate that a displacement had occurred. Reverser — the child said what could change the array back to the way it was.

All 48 children in the subtraction condition noticed the change in number, as evidenced by their surprise, their comments or their response regarding the winning status of the altered array. The vast majority of these same children also searched — presumably for the missing item. In contrast, only 25 of the 48 displacement subjects showed that they noticed the change, and none of them searched. These differences in responses suggest that young children do know that subtraction is a number-relevant transformation and that displacement is a number-irrelevant transformation. That this is a reasonable conclusion is best illustrated by quoting the children's explanations:

J.E. (3 years, 10 months) participated in the subtraction condition and after seeing the unexpected display said: *"There was three animals in the can"* (looks around). *"Took one cuz there's two now."* And, A.L. (4 years, 3 months), another subtraction subject, said: *"One's gone!"* How did that happen? *"My Ghost took it."* In contrast, E.B. (4 years, 1 month), a displacement subject who noticed the change in length, stated quite clearly that the change was irrelevant: *"Even if it's mixed up it's still three."* Why? *"Cuz, one, two, three."*

As can be seen in Table 1, *all but six subtraction* children were able to offer adequate explanations, that is, they postulated the intervention of a subtraction transformation. Further, most of these children offered ways of "fixing" the game, ways that involved adding. Fortunately, enough displacement children noticed the change, for they could then be questioned about it. As can be seen in Table 1, these children said that the change was irrelevant and most reversed the unexpected change by moving the mice closer together (or further apart if they were shown a shorter row than expected).

Since our first magic experiment, we have gone on to show that preschool children correctly classify addition as number-relevant, color change in an item as number-irrelevant and substitution of item type (e.g., a toy soldier for a green mouse) as number-irrelevant. We have run 2½-year-olds in the same design as the first experiment and obtained a similar pattern of results. And we have varied the set size of the Phase I arrays as well as the number of objects removed from either one or both of the displays. On the basis of these findings we have concluded that, by the age of three or sooner, preschoolers:

1. Know that addition and subtraction are number-relevant operations and that these operations cancel each other (Gelman, 1972a, 1972b).

2. Know that displacement, item change and color change leave unaltered the numerical value of a set (Gelman, 1972a, 1972b; Gelman & Tucker, 1975).

3. Can determine whether two arrays represent an equivalance or a nonequivalence relation (Gelman & Gallistel, 1978). And in the case of nonequivalence they can determine which array contains "more," — even if they cannot use the terms "more" and "less" (Bullock & Gelman, 1977).

4. Reason arithmetically, yet their ability is limited to those situation where they can accurately achieve a numerical representation of an array. And, in this regard, their ability to achieve an accurate numerical representation is limited to smallish set sizes, i.e., 2 - 5 items (Gelman, 1972a; Gelman & Gallistel, 1978).

5. Know that the addition of one item can be undone by the subtraction of one item. However, if addition or subtraction involves values greater than 1, they may have trouble stating the precise value that undoes the effect of addition or subtraction. Nevertheless, they solve the problem (Gelman & Gallistel, 1978). This is illustrated in the protocol from subject V.B. (age 4 years, 4 months) who participated in the subtraction condition of an experiment involving a 5-item plate and a 3-item plate in Phase 1. The 5-item plate was the winner and was changed to a 3-item plate during Phase II.

(Phase I) Why win? *"Cause there's one, two three, four, five."* Why lose? *"Cause one, two three."* (Phase II) First V.B. uncovers the 3-item plate.) Win? *"No. . . . three mouses."* Okay, which plate wins? (V.B. points to the remaining can and lifts it). Win? *"Wait! There's one, two, three."* Is that the plate that wins? *"No."* Why? *"Because it has three. It has three!!"* What happened? *"Must have disappeared."* What? *"The other mouses?"* Where did they disappear from? *"One was here and one was here."* (She points to spaces on the notransformed plate.) How many now? *"One, two three."* How many at the beginning of the game? *"There was one there, one there, one there, one there, one there."* How many? *"Five — this one is three not but before it was five.* What would you need to fix this game? *"I'm not really sure because my brother is real big and he could tell."* What do you think he would need? *"Well, I don't know — some things come back."* (Experimenter hands V.B. some objects including four mice. V.B. puts all four mice on one plate.) *"There, now there's one, two, three, four, five, six, seven! No—I'll take these* (points to two mice) *off and we'll see how many.* (V.B. removes one and counts.) *One, two, three, four, five — no, one, two three, four. Wh...there were five, right?"* Right. *"I'll put this one here* (on table), *and then we'll see how many there is now.* (V.B. takes one off and counts) *One, two, three, four, five. Five! Five."*

In Phase II, V.B. concluded that some mice had been removed and then went on to try to reinstate the original value. She did this in a series of add, count, subtract, count, etc. trials until she ended up with five items. She knew that subtractions could undo additions and vice versa. However she did not seem to be able to use a precise inverse rule. Older children and adults come to know that the addition of x can be cancelled by the subtraction of x and therefore need not go through the trial and error sequence that V.B. used. Put differently, even though V.B. knew that addition and subtraction cancel each other, she was not very good at solving a difference equation.

Let me draw your attention to V.B.'s ubiquitous use of a counting algorithm. This is a typical observation in the magic experiments, a point that leads to the next section of the paper. It appears as if young children count in order to achieve a numerical representation of an array.

The Use of Counting to Represent Number

When I say that young children can count, I do not mean to say that a child who can rattle off the first four or five count words is necessarily counting. Indeed, as we shall see, it is possible to say that a young child can count even if he or she uses a different list of count words than we use as adults. To make sense of what has just been said, it is necessary to sketch our account (Gelman & Gallistel, 1978) of counting.

We hold that successful counting involves the coordinated application of five principles. These are: (a) The one-one principle—each item in an array must be tagged with one and only one unique tag; (b) The stable-order principle—the tags used must be drawn from a stably-ordered list; (c) The cardinal principle—the last tag used for a particular count serves to represent the cardinal number of the array; (d) The abstraction principle—any set of items may be collected together for a count. It does not matter whether they are identical, three-dimensional, imagined or real. For, in principle, any set of discrete materials can be represented as the contents of a set; and (e) The order-irrelevance principle—the order in which items in a set are tagged is irrelevant. It matters not whether a given object is tagged as one, two, three, etc., so long as it is tagged but once and so long as the stable-order and cardinal principles are honored. Number words are arbitrary tags.

These counting principles capture different things about the nature of counting. The first three are principles of "how-to," the fourth is a principle of "what-to" count when applying the "how-to" principles. Finally, the fifth principle represents a combination of the "what-to" and "how-to" count principles. In summarizing our evidence on the use of the counting principles, I will first focus on the procedural, or "how-to" principles, and then move on to the remaining principles.

Evidence for the Use of the "How-To" Count Principles.

As already indicated, we have run 2½ year-olds in a magic experiment. When these children encountered a change in number in Phase II, a strange thing happened. Despite their limited ability to talk, number words started popping out of their mouths. They'd say, "2-6," we'd say "Huh? How many?" They'd say "6" and we's say "Huh. . ." I had the sense that these young children were telling us they knew they were in a number game if only I could crack their code. Since older children clearly were counting in the magic experiments they participated in, I decided to reanalyze a large subset of our protocols with a view toward determining whether our subjects were honoring the "how-to" count principles. That is, were they assigning as many unique tags as there were items to tag, were they using a stably ordered list over trials, and were they using their last tag to indicate the numerical value represented in our displays? We found that our 3- to 5-year-old subjects were remarkably good at honoring these counting principles for the set sizes involved—that is, set sizes up to as many as five. More important, we finally made sense of the two-year-old data. It turned out that some 30% of our 2-year-olds were using what we call idiosyncratic count lists like 2-6-10 or A-B-C. Moreover, they used these lists in accord with the "how-to-count" principles. Thus, a 2½-year-old child would say "2-6" when counting a 2-item array and "2-6-10" when counting a 3-item array (the one-one principle). The same child would use her list over and over again (the stable-order principle) and when asked how many items were present, repeated the last tag in her list (the cardinal principle). In the present example, the child said "10" when asked about the number in a 3-item array (the cardinal principle). Since we first reported this fact, we have been hearing from parents of 2½-year-olds that their children likewise had initial count lists that were idiosyncratic. My own son, Adam, at 2 years, 5 months had the list of 1, 2, 3, 6, 10, 11, 16 and insisted on it past his third birthday. My favorite example was brought to my attention by a member in the audience of a talk I gave. Her child's list was 1, 2, 3, 4, 5, 6, 7, H, I, J.

The fact that young children use their own lists suggests that the how-to-count principles are guiding the search for appropriate tags. Such "errors" in counting are like the "errors" made by young language learners (e.g., I runned). In the case of the language learner, we take their errors to show that the child's use of language is rule-governed and that these rules come from the child (we are not likely to hear fluent speakers of English saying *runned, footses, mouses, unthirsty*, or *2-6-10*. Further facts about the nature of counting support the idea that some basic principles guide the child's acquisition of skill at counting. First, young children spontaneously self-correct their count errors. Second, they are inclined to count without any request to do so. If we accept the idea that the counting principles are available to the child, the fact that young children count spontaneously without external motivation fits well. Likewise, the self-correction tendency is explicable. Further, the self-generated practice trials make it possible for the child to develop skill at applying the principles. Again, I quote from children to illustrate these tendencies.

M.F. (4 years, 5 months) participated in a magic experiment that involved 3-item and 5-item displays in Phase I, during which the following occurred: Why does that (the 3-item plate) lose? *"Cause it's one, two, three."* Why does that (the 5-item plate) win? *"Because one, two, three, four, five, six, Mm-m. One, two, three, four, five."* (M.F. had counted the last item twice and then, without any input from the experimenter, recounted correctly. She immediately proceeded to count again.) *"One, two, three, four. No, one, two, three, four, five. Cause it's five."*

A.B. (3 years, 6 months) participated in an undergraduate's project on language acquisition. During the course of a long interview, she encountered a display of eight items and began counting. *"One, two, three, four, eight, ten, eleben. No, try dat again. One, two three, four, five, ten, eleben.* No, *try dat again: One, two! three-ee-four, five, ten-eleben. No."* (This pattern of self-correction continued for many attempts and ended with the following count, which may or may not have been error free.) *"One, two, three, four, five, six, seven, eleven. Whew!"*

The Counting Task

Lest I give the impression that preschoolers are perfect in their application of the "how-to" counting principles, I turn to the results of a study we did to obtain videotaped records of how children 2½ - 5 years old count displays varying in set size from 2 - 19 items (where the items varied in color and were three-dimensional). Since this experiment was designed to ask the child to count a given display six times and count displays of 2, 3, 4, 5, 7, 9, 11, and 19 items, we assumed that we should do things to mitigate against boredom and flagging interests on the part of our subjects — who came for a day-care program serving children from a heterogenous population. The best trick we knew of was to engage the child in a three-way conversation with a puppet who at various times feigned ignorance and asked the child to help him, the puppet, understand. What did we find? The 3- to 5-year-old data are presented here, starting with the use of the one-one principle.[1]

The one-one principle. The crudest index of emerging counting skills is whether or not children use as many tags (unique or not) as there are items to be counted. This they generally do, as seen in Figure 1 where the tendency of children who counted a given set size to use $N = X$ tags or $N = X \pm 1$ tags when counting various set sizes is shown. The top panel shows the extent to which the number of tags used corresponded perfectly with set size. The bottom panel shows the extent to which children used either the exact number of tags or one too many or one too few tags. Note that a considerable part of the failure to be perfect here is due to a tendency to overshoot or undershoot by one tag. So if we consider 4- and 5-year-olds under the perfect condition, we see that they start to err around set size four or five. When we look down at the bottom panel, we see that these errors are almost all of the $\pm$ 1 kinds. The size of this comparison is even more noteworthy for the 3-year-olds, the majority of whom when attempting a count use $N \pm$ tags.

To me these data suggested that the children were applying the one-one principle and that most of their errors were due to performance demands. To check this possibility out, we conducted detailed analyses of error types and found that the error types that occurred were consistent with out hypothesis. Children hardly ever repeated a tag or randomly skipped around as they counted. Instead they systematically proceeded to tag adjoining pairs of items. Thus, when arrays were linear, this means they typically started at one end and continued through to the other. Most errors that did occur involved a tendency to double count on an item and to skip an item. Further, the children had some difficulty in coordinating the complex motor activity involved in counting. In particular, the younger children had some trouble getting together the acts of pointing and talking at the beginning of a count; likewise, they had trouble stopping these acts together at the end of the count. Systematic sources of error like these fit well with the idea that the child still has to practice at counting even though he or she knows (not necessarily consciously) what it is about.

Stable-ordering. There is little to say here except that the children were remarkably consistent in their use of a list across trials. More than 90% of the 4- and 5-year-olds and 80% of the 3-year-olds used the same list on all of their trials, regardless of set size. Again, we allowed for the use of idiosyncratic lists, and again we saw them — their appearance seems to be dependent on age and set size.

The cardinal principle. The percentage of children in each of the age groups who applied the cardinal principle is shown as a function of set size in Figure 2. There is an obvious effect of set size and age. The younger child, the less the tendency to repeat the last tag or indicate the cardinal number of a set. Likewise, the larger the set size, the less the tendency for the child to report the cardinal value of a set.

Thus, young children can honor the principles of the counting procedure. Still, they do err in their application of the how-to-count principles. That they do err provides one clue as to why the young child's ability to reason arithmetically is limited to small set sizes. Since their ability to count correctly — and therefore their ability to represent accurately the numerical value of a set — is limited to small sets, they cannot help but have trouble keeping track of larger set sizes when reasoning about them.

[1]The two-year-old results are reported in Gelman and Gallistel (1978, pp. 131-135). The need to treat these data separately makes it hard to present them along with those from the other three age groups. The basic results were that two-year-olds showed evidence of following components of the counting principle, and they were better able to count if they used their own idiosyncratic count lists.

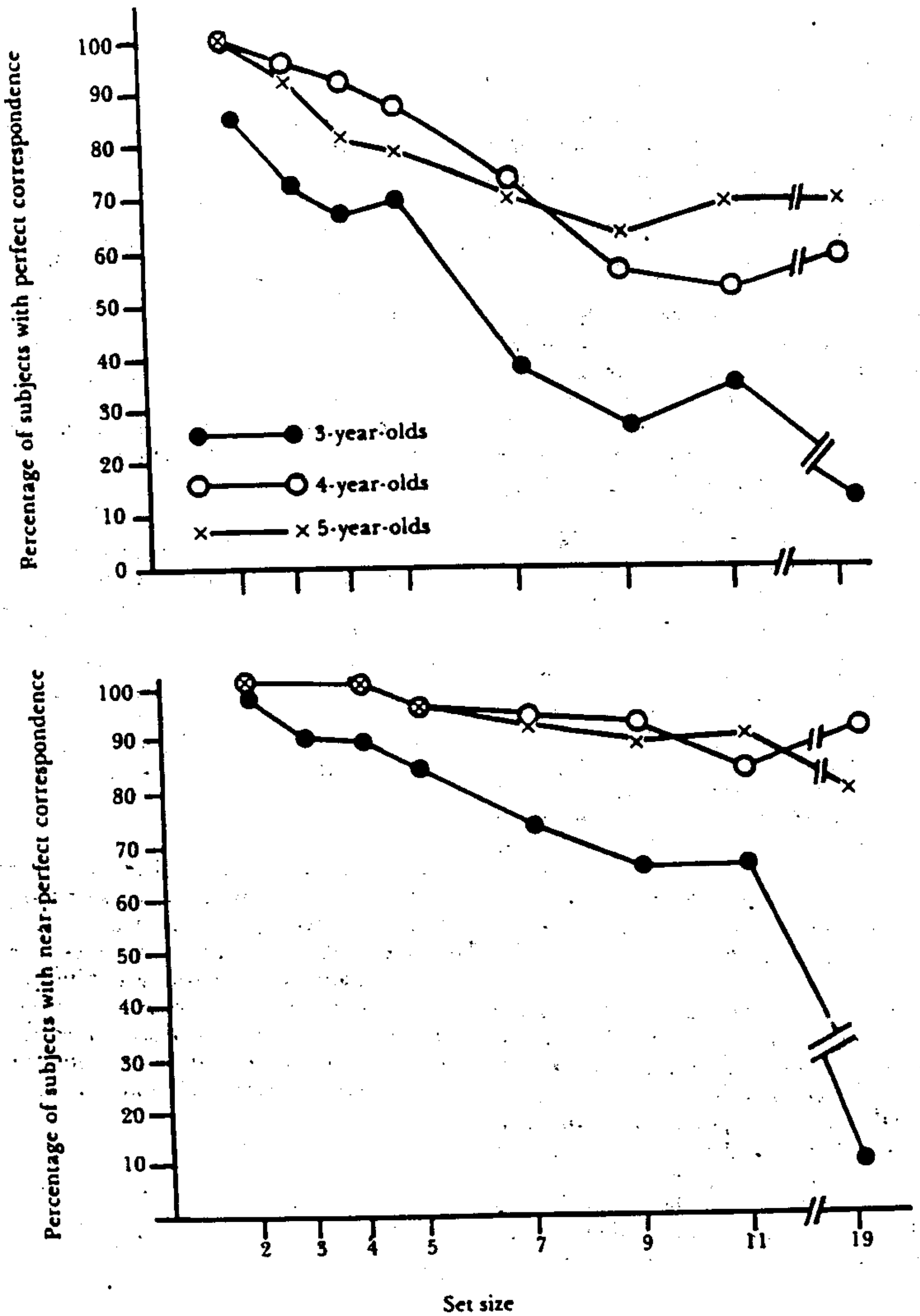

Figure 1. Percentage of children in counting experiment who achieved perfect or nearly perfect correspondence between set size and number of tags used. Upper graph shows results for those who used exactly N tags for a set of N items; lower graph also includes those who used N ± 1 tags. From Gelman and Gallistel (1978, p. 109); used here with permission.

The other principles. Do preschoolers know that item type is irrelevant to the application of the counting procedure (the abstraction principle)? I believe that they are remarkably indifferent to item type and cite several sources of evidence. First, in our various experiments we have varied types of items, including a two-dimensional versus three-dimensional comparison and a homogeneous versus heterogenous comparison. We see little, if any, effect of these variations (Gelman & Gallistel, 1978). Second, there are the results of an experiment on whether preschool children might refuse to collect together people and inanimate physical objects for a count (cf. Keil, 1977). As a test of this hypothesis, I recently asked 3-year-olds (N = 19) and 4-year-olds (N = 21) to count all of the things in a room. The questions were (a) whether the children would spontaneously include themselves and/or the experimenter in the count; and (b) if not, would they, once probed, continue their count to include people in the collection being counted. The results were that 20% and 16%, respectively, of the 3- and 4-year-olds spontaneously included people in their count. Another 53% and 68%, respectively, did so when probed. That is when asked, "what about us?" they continued their count. Therefore, all but 27% and 16% of the 3- and 4-year-olds were willing to put together the inanimate items in a room with people for purposes of a count. Thus, although there was some tendency to separate animate and inanimate objects for the purposes of a count, it hardly constituted a strong restriction on the definition of a countable collection. Whether young children will count imaginary things or events remains a question for further research. Still, I am struck by the general willingness of preschoolers to count almost anything.

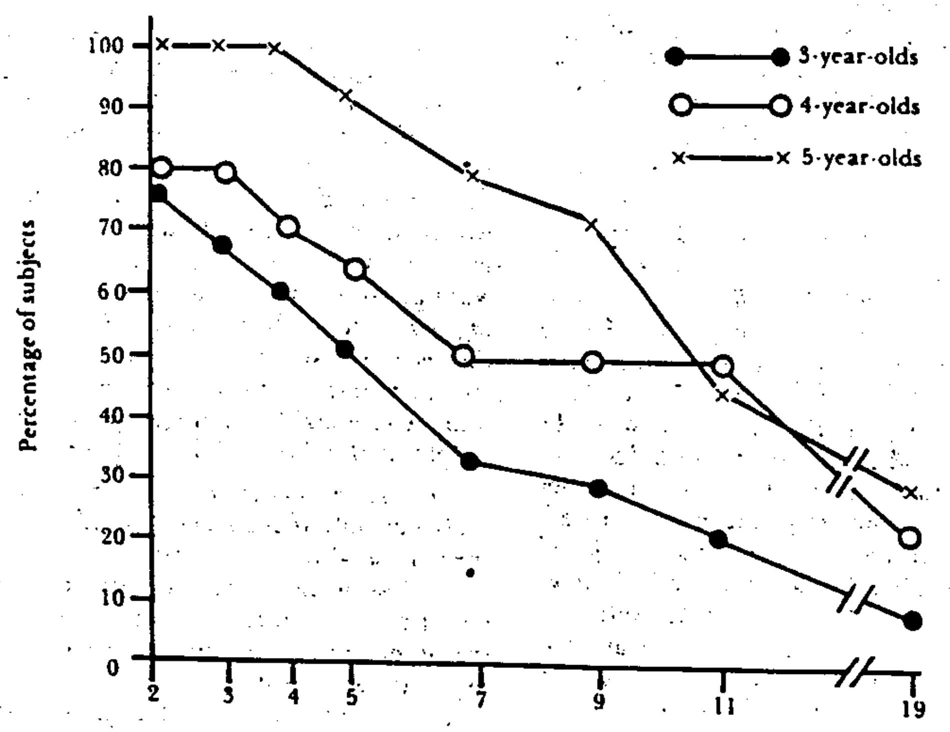

Figure 2. Percentage of children employing the cardinal principle at the end of a count as function of set size. Most children scored on the basis of their having: repeated the last tag in their enumeration; counted on one trial and stating the cardinal number on a later trial; or simply stating the cardinal number. See Gelman and Gallistel (1978, pp. 122-124) for a discussion of scoring criteria. From Gelman and Gallistel (1978, p. 124); used here with permission.

It is not being claimed that there are not stimulus variables that will influence a child's willingness to count. Indeed, I expected that conditions which make it hard for young children to touch and/or move objects could influence their willingness to count. The idea here was that it would be harder for children to keep separate items already tagged from those that have yet to be tagged and that the children would recognize this. This hypothesis was confirmed in a recent experiment. We compared the ability of children to count three-dimensional materials they could touch and move with their ability to count the same three-dimensional materials under a plexiglass cover. Overall, that is across all set sizes, we saw a notable drop in a child's performance when they were asked to count objects that were under plexiglass. Three-year-olds (N = 18) were able to apply all three count principles on 55% of their trials with uncovered set sizes varying from 2 - 19. The comparable figure is 31% when the plexiglass cover was in place. For the 4-year-olds (N = 18) the overall trials correct were 82% and 68%, respectively. As expected, many children protested that they wanted to feel and/or touch the toys.

When we looked at performance as a function of set size, there was a clear interaction between condition, age, and set size. Three-year-olds were at a disadvantage in the plexiglass condition for all set sizes. Although 87% of their trials on set size 3 were correct when they worked with uncovered three-dimensional objects, only 49% of their trials were correct when they worked with *the same 3-item display* under plexiglass. In contrast, 4-year-olds did about as well under both conditions (almost perfectly) for set sizes 2, 3, 4 and 5. It was only when set sizes reached 7 that the covered display gave them trouble. With a set size of 7 items that were not covered, the older children correctly applied the one-one principle; only 75% of the covered trials were negotiated as well.

Thus, it seems that stimulus variables affect counting to the extent that they interfere with the child's tendency to point to, touch, and move objects. The younger the child, the more they need to point to and/or touch what they count. However, this variable is orthogonal to the variable of item type. We see young children willing to collect together a rather large range of stimulus types for a given count. Presumably, their application of the abstraction principle is not as wide as that of an adult. Nevertheless, there does not seem to be an extreme restriction on their definition of countables.

What about the order-irrelevance principle? We have both *weak* and *strong* data. The weak data come from asking whether children who repeatedly count a set always try to assign the same tag to the same object — even when the display is rearranged. The answer is no. The young child seems indifferent as to whether a particular tag goes with a particular item. The strong data come from an experiment designed to require children to reassign tags systematically to different objects in a given display. We presented two tasks to 3- and 5- year-old children (Gelman & Gallistel, 1978). Only one — a modified counting task — will be described here. After a child counted a linear array of 4 or 5 small heterogeneous items, we asked them to participate in a demonstration of tricks to a puppet. The trick is to count the items so that the second from their left (a baby or a chair) comes out *the one*. This done, they were asked to make the same object be *the two*, then *the three*, and so forth — thereby demonstrating that it does not matter much which item is tagged with a given tag. Furthermore, they also have shown a willingness to change the order in which they tag given items. Remember they have to tag the second-position item as *one, two, three.*, etc. Incidentally, to succeed here is to show some considerable strategic planning for a novel task (Merkin & Gelman, Note 1).

To be scored as perfect on this task, the child must not only make x *the one, the two*, etc., they must also count correctly on all trials. Still, 75% of the 5-year-olds turned in perfect performances, indicating that they understand the order-irrelevance principle well enough to negotiate an unusual task. The younger the group, the poorer the performance. And not too surprisingly, the poorer the child was at simple counting, the poorer they did on this task. Thus, if we apply our hard criterion for judging the child's ability to honor the order-irrelevance principle, the vast majority of our five-year-olds succeed. When we apply the weak criterion, even our three-year-olds seem not to care that on one trial a doll is treated as *the one*, and on another trial that a chair is treated as *the one*.

Summary

I believe the case has been made for the view that preschoolers can and do count and that their skill at counting is related to their ability to reason numerically. Attention is drawn to two facts: First, young children count even though they may not be told to and without the benefit of a curriculum. Second, when preschoolers are tested in the magic paradigm, they reveal an ability to treat addition and subtraction as number-relevant and displacement and item change as number-irrelevant transformations. Thus, the conservation task under-estimates their operational understanding of number. I shall return to these points. But first, some findings from research with somewhat older children.

An Inquiry into the Understanding of Infinity

I want to report briefly on the results that Diane Evans and I have on the developing understanding of infinity in children aged 5 - 9 years who represent a lower-middle- to middle-middle-class population. These children were not taught about the concept of infinity. So our experiment is another case of asking what children will do when we give them the opportunity to think about something they have not been taught. The infinity questions were but one set of questions we put to these children. We also asked them about zero, negative numbers, the concept of "forever," what happens when a ball is dropped into a bottomless pit, and so on. The infinity questions started out with Evans asking the children to name the biggest number they could think of. They were then asked about the effects of the repeated additions of one more item, e.g., "What happens if I add one to the number you gave me?" We sought to determine whether and when children: (a) appreciate that the iteration of one more can go on forever and the effect will be yet a larger number (Closure), and (b) know that there is no biggest number. We are able to classify reliably a given child's protocol at one of three levels:

Level I: Finite and small. Children classified as Level-I responders often claimed they could not add one to the number they said was the biggest they could think of. Their ability to counted seemed limited (say to 20 or 30), and the numbers they mentioned as "the biggest" were usually less than 100 or made-up numbers, e.g., "twenty-eight-thirty-one." The following protocol illustrates this level of response.

M.C. (5 years, 11 months). What's the biggest number you can think of? *"Fifty."* Is that the biggest number that could ever be? *"No."* What is? *"Twenty-twenty-nine."* What would happen if we added one to that? *"Fifty."* Is that the biggest number? *"No. . . Forty-thirty-three."* What if we added one to that? *"Forty."* Is that bigger than Forty-thirty-three? *"No. . . Two . . . One-hundred-thirty."* What would we do to get a bigger number? What is it? *"Thirty-twenty-ten."* Is that the biggest number? *Yes."* Could we add one more to it? *"No."*

Level II: Finite and large. Children classified as Level II typically mentioned very large numbers (e.g., a million) in response to the question about the largest number they could think of. They say that one can keep adding and getting yet larger numbers. But paradoxically they insist that there is a point at which one may no longer add one or that there is nevertheless a largest number, as illustrated in B. G.'s protocol.

B.G. (6 years, 11 months). What's the biggest number you can think of? *"A million."* Is that the biggest number there could ever be? *"No . . . It goes all the way up to a trillion?"* Is that the biggest number? *"Yes."* What if we add one to a trillion? *"A billion."* Is that the biggest number? *"I don't know."* What if you add one to a billion? *"A billion and one."* One more? *"A billion and two."* Add one more? *"A billion and three."* Is that the biggest number? *"Yes."* What if you add one more? *"A billion and four."* Is that bigger? *"Yes."* (The conversation continued with B. G. saying that everytime *E* asked if one could add that one could. Nevertheless he also said there was a biggest number. The session ended as follows:) Can we find a number that is bigger than the biggest umber? *"No."* What if we added one to the biggest? *"You'd have to start again."* Wouldn't you get a bigger number? *"No."* What would happen? *"I don't know."* Do we ever have to stop adding one? *"Maybe... My sister told me the numbers never stop."* Do you believe that? *"No. It sound funny."* Why? *"I think numbers stop."*

Level III: Infinite. Children at this level start out with very large numbers, say that one can keep adding, and that the count numbers are unbounded, i.e., there is no largest number. The following is from a conversation with a second grader.

S.D. (7 years, 10 months). What's the biggest number you can think of? *"A billion."* Is that the biggest number? *"No."* What is? *"I don't know."* Why don't you know? *"Numbers never end."* What if someone told you that a googol is the biggest number? *"I'd say that numbers don't end."* Could you prove it? *"No."* What if you added one to a googol? *"A googol and one."* What if someone said that a googol and one must be the largest number? *"No."* How can you prove it? *"Add one more."* If someone tells you that there is a biggest number, what would you say? *"No, there isn't, because numbers never end and there's always a bigger number."*

None of our kindergarten subjects in the experiment proper were scored as Level-III responders. Half, however, of them (N = 18) were scored at Level II. More surprising to us was the finding that slightly more than half of the children in Grade 1 (N = 20) and Grade 2 (N = 19) *were* scored as Level-III responders. And 86% (N = 22) of the third graders were scored at Level III. Thus, children come to know that, in principle, an iterative process is involved in the successive generation of numbers and thus that there is no final number. How does this happen? These concepts are not taught in the school. What then is the source? I can think of three possibilities.

The first is that someone other than a teacher, a parent or a brother or even a classmate, stood in as instructor. In a pilot study, we soon discovered that our questions to some second graders became the topic of conversation when the children were on the playground. Thus, so-called naive subjects came to us ready to tell us that they knew the answers to the questions we would pose! The second source is ourselves. In many cases we had the sense that our subjects "caught on" as we questioned them. In the last protocol, note how the experimenter led the child to an understanding of a proof — at least as regards the fact that there is no largest number. The final possible source of input is the child himself. The decision to question children about "the largest number" followed after a variety of parents reported that their young child has asked them if there was a largest number. This question was often posed during a period where the child seemed preoccupied with counting to a large number, such as a thousand or even a million. I remember a five-year-old who told me that she had been trying to count to a million, had been at it for two days and said it would still take her a long time to reach her goal!

Note what seems to be involved here. Children, either on their own or with questioning from someone else, become involved in a thought experiment and then reach an induction about some properties of the number system. Our findings suggest that first- and second-grade children are "ready" to engage a thought experiment about infinity. I say "ready" because they, unlike some of our younger subjects, follow the line of our questioning about continued iteration through to its consequence — this being that there is no largest number. There is much to understand about these findings, not the least of which is why children come to know about infinity even though it is not taught in school.

Issues and Implications

Proceed with Caution: Or Games are Games

We find that young children know a great deal about numbers. I must emphasize what is a general feature of our research. We ask children to think about questions they are not asked in their schools. For school-aged children there is the possibility that they take our questions as puzzles and therefore think they are in a game setting. In our work with preschoolers we use games *by design* and imbed our research questions in these games.

Recall the use of the magic show to assess arithmetic abilities and the use of puppets for children to talk to, to teach, or to demonstrate counting tricks. We provide as many helping hands as possible. This is fine when one is interested in describing underlying capacities. But capacities that show under our conditions might very well be recalcitrant to showing under a wide range of conditions (cf. Gelman & Gallistel, 1978). That is, it is still a question as to whether such abilties will generalize with relative ease.

There is another feature of our work that sets it apart from many school situations. We assess a child's ability in a one-on-one setting where the experimenter is prepared to adjust the line of questioning so as to maximize the likelihood of a child succeeding or determining why a child errs. Again the child is getting an unusual amount of support and guidance. If the question is what capacity, if any, does the child have for dealing with a particular conceptual domain, then it is encumbent upon the researcher to use every resource available to tap that ability. What is not clear to me is whether a similar strategy is what educators should rely upon. Indeed, I can think of many reasons not to do the same.

As I see it, educational programs have as their goal the development of knowledge systems that can be accessed by the child on his or her own. The hope is that eventually children will *not* be dependent on the probes of adults or other supports to begin to think about a problem and how to solve that problem (cf. Vygotsky, 1978). The goal is that a child's understanding of a concept will generalize across a wide range of settings. Thus, it is necessary to determine the kinds of educational programs that encourage generalization of a child's nascent knowledge of arithmetic to a wide range of tasks — especially those tasks that require a child to think of what to do without any probes from others.

Talk about Readiness

It is commonplace to read that there is not much point in trying to teach arithmetic before a child passes the number conservation task (Piaget, I believe, would not say this). The implicit assumption in such an advisory is that the child who fails to conserve has no number concepts. This in turn leads to the idea that the nonconserver is not *ready* for arithmetic instruction. There are at least two problems with this argument.

First, young children do know something about number concepts even if they cannot conserve as assessed by the Piagetian task. Second, it's not clear what role conservation status plays in an outline of the notions of readiness where the notion refers to a curriculum plan for the future. Should we teach children to conserve? There is a case to be made for the view that this is a waste of a teacher's time. For, it seems that all normal children, independent of schooling, come to conserve. Put differently, the ability seems not to require structured school inputs. Since it seems that there are other subjects that do require structured inputs, it seems sensible to focus on the question of what these structured inputs must be.

It still remains to be determined what the ability to conserve number is related to. Gallistel and I (Gelman & Gallistel, 1978) suggest that conservation may index the onset of an ability to reason arithmetically about undetermined values. The idea is that unlike preschoolers, elementary school-aged children are able to reason algebraically, to think about numbers as a variable and determine the effects of arithmetic operations without thinking about a specific number. If we are right, it might turn out that conservation indexes the beginning development of higher-order, arithmetic-reasoning abilties and *not* the beginning understanding of numbers. In any case, it hardly can be maintained that the child who fails on a number conservation test knows nothing about numbers.

Universal Concepts and What to Teach

The idea that there are some universal number concepts has been touched upon already in my remarks on conservation. In the case of conservation, it seems that all children develop the ability to conserve number despite wide variations in cultural settings and the availability of schooling. I believe that the kind of arithmetic abilities we grant preschoolers are likewise universal. First, it appears that most cultures use a counting procedure. Zaslavsky (1973) finds that African groups presumed to lack the ability to count can in fact count. Conclusions to the contrary have been based, in part, on the presumption that one must use the conventional count word sequence as the tagging sequence. Saxe's (Note 2) finding that people in a village of Papua, New Guinea use successive parts of their bodies as the counting tags provide a marvelous example of the ability to count without using counting words.

Second, Ginsburg and his collaborators find that children in the Ivory Coast use counting to do simple problems in addition whether or not they attend school and whether or not they live in a trading society. Like us, Posner and Ginsburg (Note 3) conclude "children can learn to do arithmetic without any formal tutelage." Ginsburg's (Note 4) work with lower-class preschoolers in the Baltimore and Washington, D.C. areas supports his view that there are "natural" arithmetic abilties that develop without the support of a schooling environment. When tested with tasks that assessed their understanding of "more," their ability to count and to do addition using a counting algorithm, these children showed the same pattern of errors as did middle-class children. The implication here is that a common error pattern reflects a common underlying capacity. Add to all of these findings the fact that Premack has had little, if any, success in teaching chimpanzees to count or compare discrete quantities (Premack, 1976; Premack, Woodruff, & Kennel, 1978), and one has a hard time avoiding a surprising and exciting generalization: Counting and its use as an algorithm might very well be a universal ability of normal people.

This raises a major question. What, if any, role do such "natural" abilities play in the acquisition of school subjects or knowledge that seems to be dependent on specific instruction? To be sure, the absence of these abilities could be a serious obstacle to the teaching of arithmetic and other kinds of mathematics. And I am currently assessing their status in retarded children with a view to determining the source of their difficulties with arithmetic. However, the opposite need not be true; we cannot assume that the naturally developing abilities guarantee that the teaching of related abilities will be trivial.

To make my point, let me draw on another case of a natural, universal ability, the understanding of and production of speech. It seems fair to say that every normal child can and will acquire language. What is more, the acquisition will go fast enough for us to assume that the child is reasonably fluent in the mother tongue of his or her environment by the age of four. And, although linguists differ on the particulars regarding a description of this capacity, there is little dispute about certain facts. Language learning, although dependent on a supporting environment, seems to be able to proceed without structured lesson plans (Feldman, Goldin-Meadow, & Gleitman, 1978; Newport, Gleitman, & Gleitman, 1977). Second, language ability develops very quickly and is very complex. Still, to my knowledge there is no massive movement towards the abandonment of grammar classes. Why not? Educators are wise and recognize a basic distinction between the ability to talk and the ability to access the structure of the language that is spoken. Grammar classes are designed to teach children to recognize the rules that are inherent in their use. Being able to speak does not guarantee the spontaneous development of an awareness of the rules of grammar that govern the regularity of speech production and comprehension. What is more, mastering one's grammar lessons is not sufficient to make one able to do linguistics, i.e., derive the nature of the structures that most accurately capture the facts of our linguistic abilities. This is the task that professional linguists set for themselves, individuals who have studied long and hard to do linguistics.

I believe a similar line of argument holds regarding the teaching of mathematics. Even if I am correct in assuming that there are universal arithmetic abilities, it does not follow that the teaching of mathematics will be trivial. It is quite reasonable to suppose that early instruction serves to teach children to access the principles that guide their early arithmetic solutions. And even if this is done well by mathematics teachers, I would be cautious about assuming that then the children will be able to "do" arithmetic. Like linguists, mathematicians need to master a great deal about the formal disciplines of mathematics before starting out on their own "to do mathematics."

On Teaching Children at Their Level

I have cautioned against a heavy reliance on Piagetian norms for the purposes of defining the introduction of arithmetic into the curriculum. Still, I do not for a moment want to give the impression that we can forget about a general implication of Piaget's work for education: that instruction should be geared to the child's level of cognitive development. My most general point is that we are just beginning to understand what it is young children can and cannot do, and, as such, it is very hard to meet this general tenent. Further, we need to determine what kinds of concepts need structured inputs and what these inputs should look like — especially if we have a goal of teaching that which will generalize.

To illustrate that we still are far from knowing what it is children can do, I return to one result of my own research which puzzles me. This is that there seems to be a crucial role of thought experiments in the development of mathematical ideas (cf. Kuhn on scientific ideas, 1977). Our studies of infinity asked children to think about the nature of natural numbers. Most of the children had not done so until we coached them. Yet, in a very short order they were onto a discovery. Some children do this on their own, and many do not. What this all means is not clear. But, I am confident that the answer to this puzzle will yield an important message for those who think about the nature of conceptual change and how it can be fostered — at least in the domain of mathematics.

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