The Child's Representation of Number: A Multidimensional Scaling Analysis

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MILLER, KEVIN, and GELMAN, ROCHEL. *The Child's Representation of Number: A Multidimensional Scaling Analysis*. CHILD DEVELOPMENT, 1983, **54**, 1470–1479. In order to describe developments in children's conceptions of number, judgments of similarities between numbers were solicited from children in grades kindergarten, 3, and 6 as well as from adults. Analysis of the resulting data by clustering and nonmetric multidimensional scaling techniques suggested that children become sensitive to an expanding set of numerical relations during this period, although even kindergartners appear to understand the importance of magnitude as a basis for judging similarity between numbers. Results implicate the acquisition of numerical skills and operations such as counting, addition, and multiplication in this broadening of the concept of number. A second study was conducted with a group of kindergartners in which letters replaced numbers as stimuli, and an explicit criterion based on closeness between stimuli in the alphabet was given to subjects as the basis for judging similarity. These results suggest that the number-similarity judgments of kindergartners, and to a lesser extent of older children, are based on counting distance. Implications of this research for the distinction between defining features of number (such as one-to-one correspondence) and numerical operations in children's reasoning about numbers are discussed.

Developmental research on children's understanding of numbers has focused on the comprehension of basic logical concepts that can be used to define number. Russell (1903; Whitehead & Russell, 1910–13) noted that two sets are equal in number if and only if their members can be placed in one-to-one alignment. Piaget (1965) demonstrated that preschool children fail to respect this principle of one-to-one correspondence, frequently asserting that the numerosity of a row of objects becomes greater as the length of the array increases. Children who understand the concept of one-to-one correspondence are said to have the "number concept," and further development is described as the learning of applications for this concept. As Piaget (1965, pp. 161, 241) notes, "Additive and multiplicative operations are already implied in number as such, since a number is an additive union of units, and one-one correspondence between two sets entails multiplication. . . . Addition and multiplication of classes, relations and numbers are implicit in the construction of every class, every relation and every number."

Perhaps because of this belief that one-to-one correspondence defines the concept of number for the child as well as for the mathematician, the possibility that learning new arithmetical operations alters children's conceptions of numbers has not been investigated. Development beyond the point at which children conserve number may consist merely of the learning of new applications for this number concept. Alternatively, mastering new operations may

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alter in some manner children's conception of what numbers are. Exploration of the features of number that children represent at different ages is necessary to distinguish between these alternative views of the nature of number development. A prerequisite to such an exploration is some procedure for producing a tangible representation of the characteristics that children at different ages see as the basic features of numbers.

One approach to studying which features of number are represented was employed by Shepard, Kilpatric, and Cunningham (1975). Shepard et al. asked adults to rate the similarity between pairs of numbers presented under a variety of conditions. The resulting similarity matrices were analyzed by a nonmetric multidimensional scaling algorithm and later reanalyzed by a nonhierarchical clustering procedure (Shepard & Arabie, 1979). Judging integers as "abstract concepts," subjects appeared to be sensitive to a variety of numerical relations. In addition to magnitude, these included multiplicative relations such as even/odd numbers and powers of two and three. These results suggest that the number concept of adults includes a variety of relations in addition to such defining features as one-to-one correspondence. The present study was undertaken in an attempt to clarify the developmental course by which children become sensitive to these features and the relation that these enlargements of the concept of number may have to the acquisition of numerical operations.

Experiment 1

Method

Design.—Shepard et al. asked adults to rate the similarity between all of the 45 distinct pairs of single-digit integers (0–9). Fearing that this might be an inappropriately complex judgmental task for young children, we chose to adopt the method of triads (e.g., Levelt, van de Geer, & Plomp, 1966), requiring subjects simply to choose the least similar and most similar pairs from a set of three stimuli. The triadic judgment task was successfully employed in an earlier developmental study by Arabie, Kosslyn, and Nelson (1975), so we were confident that the procedure was appropriate for young children.

While the use of triads results in a simpler judgmental task, this simplification occurs at the expense of greatly increasing the number of unique stimulus combinations for subjects to judge (a total of 120 unique triads from the 10 single-digit integers). In order to decrease the number of triads presented to each subject, we created two balanced incomplete sets of 30 triads each, in which each pair of integers was presented twice, with no triads appearing more than once either within or between the two sets. This procedure resembles that employed by Levelt et al. (1966) in response to a similar problem. Ball (1914) provides a more extensive discussion of this problem (known as "Kirkman's schoolgirls problem") and procedures for producing such balanced but incomplete subsets. Presenting the triads in a subset in either increasing or decreasing order (e.g., "3,4,5" vs. "5,4,3") produced two additional balanced incomplete subsets of triads. In addition to the 30 experimental triads in each set, a randomly chosen group of 10 triads was repeated (five in increasing order, five in decreasing order) to permit evaluation of consistency of judgments across repetitions and changes in the order in which the numbers were presented. Adult subjects judged a set of 80 triads each, consisting of both sets presented to the children.

Subjects.—Subjects were 36 children from Philadelphia and an adjacent urban suburb, 12 (six boys and six girls) in each of grades kindergarten (mean age 5-8), three (mean age 8-9), and six (mean age 12-3). A group of six adult subjects (three males and three females) from the graduate students and faculty of the psychology department of the University of Pennsylvania also completed a longer version of the task administered to the children.

Apparatus.—All numbers consisted of Chartpak, Futura Demi-Bold, 48-point press-on dry transfers. Because pilot testing indicated that kindergartners had difficulty determining all three pairs present when triads were presented in a row, the following apparatus was employed: The numbers 0–9 were placed on three cardboard wheels 12 cm in diameter. These three number wheels were then placed on a plywood backing so that their centers formed an equilateral triangle. The apparatus was covered with heavy paper with only a 1.2 × 1.5-cm window permitting vision of one number on each wheel. An equilateral triangle with sides of 11.6 cm was drawn between the windows, so that a line connected each pair of stimuli.

The materials used with the adults consisted of standard 7.62 × 12.7-cm (3 × 5-

inch) index cards produced with the same press-on dry transfers used with the children. Each card contained one triad presented in a row with 2.5 cm between integers.

Procedure.—Subjects were told that the task involved judging the similarity between some numbers. A brief pretraining was given in which children judged the similarity of triads from the set "Mother," "Father," "Sister," and "Brother." In order to avoid implying that answers on the main task should involve numerical magnitude, subjects were asked to determine which two were the "most closely related or the most similar to each other." Whichever pair children picked was praised as a good answer, but the experimenter added that they also could have picked another pair (naming one of the others), giving a reason for picking the other pair (e.g., "Because they're both adults" or "Because they're both boys or males").

Following this pretraining, subjects were presented with the experimental apparatus and told that the experimenter was interested in their opinions about what numbers are similar to each other. Subjects were presented with the first triad and told that the lines on the apparatus would show them three pairs of numbers, and were asked to: (*a*) name the three pairs of numbers, (*b*) say which two numbers are the most closely related to each other (or the most similar), and (*c*) say which two numbers are the least closely related to each other. On one out of every five trials, children were asked to explain their answers, and their responses were written down verbatim by the experimenter.

Results

Comparison of similarity matrices.—The consistency of judgments across repetitions was high, with reliability of 92% or better at all age levels and no difference obtained in consistency between increasing (93%) and decreasing (91%) orders. Proximity matrices for the judged similarity between integers were calculated at each age level by adding 2 to an index for each pair picked as "most closely related," adding 1 for the pair not picked, and adding 0 for those picked as "least closely related." The resulting matrix provides an ordinal index of the relative similarity between the different integers as determined by our subjects.

The relation between similarity judgments made by the different age groups was assessed using procedures developed by Hubert (1978, 1979; Schultz & Hubert, 1976) that avoid problems confronting direct correlation of similarity matrices (Carroll & Arabie, 1980). Hubert (1978) described procedures for determining a conservative approximation of the probability of achieving a given concordance between two matrices based on permuting the rows and columns of the matrices being compared. The particular concordance statistic employed here corresponds to the Pearson product-moment correlation coefficient, although statistical significance was assessed using Hubert's procedures. A matrix of concordances between similarity judgments at different age levels is presented in Table 1. While the judgments of our children subjects were re-

TABLE 1

CONCORDANCE BETWEEN NUMBER-SIMILARITY JUDGMENTS

	Kindergarten	Third Grade	Sixth Grade	Adults
Kindergarten				
Third grade	.987**			
Sixth grade	.776*	.814*		
Adults	.435	.465	.776*	
Kindergarten letter-similarity judgments[a]	.934**	.930**	.679*	.373

NOTE.—The concordance statistic is identical to the Pearson product-moment correlation coefficient. Statistical significance was assessed following Hubert's (1979) generalized concordance procedures.

[a] The Kindergarten letter-similarity judgments were collected in Experiment 2 and will be discussed in detail during presentation of that study.

* $p < .05$.

** $p < .025$.

lated to each other, the judgments of adults showed a significant level of concordance only with the sixth graders. Given the high degree of concordance between the kindergarten and third-grade data, these results suggest that there may be an adult pattern of number representation different from that of children (at least of kindergartners and third graders), with sixth graders perhaps demonstrating some combination of the two patterns.

Multidimensional scaling analysis.—The proximity matrices were then analyzed by the KYST-2A (Kruskal, Young, & Seery, Note 1) algorithm for the nonmetric multidimensional scaling (MDS) procedure devised by Shepard (1962a, 1962b) and extended by Kruskal (1964a, 1964b). This program produces a spatial representation of similarity data such that short distances between stimuli correspond to judgments that stimuli are highly similar to each other, with more separated stimuli having been judged less closely related. The nonmetric MDS procedures used here produce a configuration of points subject to the constraint that any rigid rotation of the resulting configuration is permitted. Since the current data are ordinal in nature, the present analysis specified that the relation between distance and judged similarity need only be monotonic. The KYST-2A algorithm proceeds by minimizing a measure of badness-of-fit ("stress") of this (monotonic) relation between judged proximity and distance in the program's output. When further iterative changes do not reduce the stress function, the program terminates and the stress of the resulting scaling solution provides an index for the badness-of-fit of the monotonic relation between judged similarity in the data and distance between stimuli in the solution.

Because of the iterative nature of the KYST-2A algorithm, the possibility exists that the program may terminate at a minimum value of stress that is merely local rather than the globally lowest value. To increase the likelihood of obtaining the globally minimum stress, it is necessary to repeat the analysis from a number of different random starting configurations. Each similarity matrix was scaled from 12 initial configurations in addition to the "TORSCA" option that utilizes a preiteration procedure to find a favorable initial configuration prior to the actual scaling. The Primary Approach to Ties and Stress Formula One were used as options to control the MDS analysis. Because of the small number of stimuli used in the present study, only two-dimensional solutions were obtained.

Final stress values for the KYST-2A analysis range from .065 for the kindergarten data to .142 for the adult data. These values can be compared with Klahr's (1969) Monte Carlo evaluation of stress values obtained from an analysis of random data sets using a precursor to KYST-2A. The final stress values obtained in the present study were all less than those achieved on 95% or more of trials by Klahr for 10 data points and two dimensions, suggesting that the final configurations obtained reflect some systematic feature of the data other than random noise. Klahr's study provides only a suggestive rather than a definitive comparison, however, since his procedure differed significantly from the present analysis. In particular, Klahr used a single initial configuration, while the present study used several random starting configurations. Problems in the application of Klahr's Monte Carlo study are discussed by Arabie (1973), although Arabie concluded that the Klahr study provides the best available data for evaluating the probability that an obtained stress value does not reflect an underlying pattern in the data.

Nonhierarchical clustering analysis.—In addition to having a spatial representation of our subjects' number similarity judgments, we were specifically interested in the numerical relations emphasized at different ages. A nonhierarchical clustering model (ADCLUS) developed by Shepard and Arabie (1979) describes proximity data in terms of a set of overlapping clusters, with each cluster contributing a constant weight to the similarity between its members. In contrast to prior hierarchical clustering procedures (e.g., Johnson, 1967), the ADCLUS model permits the existence of overlapping clusters. Thus, for example, the existence of a cluster such as "1,2" based on magnitude or adjacency in the number series would not preclude finding an (overlapping) cluster such as "2,8" based on some other criterion. A related individual-differences model called INDCLUS has been developed (Carroll & Arabie, Note 2, Note 3), which assumes that different subjects or groups of subjects will share a common set of clusters but may give comparatively different emphasis to those clusters in making judgments. Differing emphasis on clusters is reflected in differences between groups of subjects in cluster *weight*, a measure of the extent to which membership in a cluster leads to an increase in judged similarity among the members of the cluster. The program for fitting this individual differences model is also called INDCLUS and is a generalization of the earlier MAPCLUS algorithm (Arabie & Carroll, 1980).

The MDS solution for the kindergartners' judgments is presented in Figure 1, and a classification of their explanations is given in Table 2. Clusters found in the INDCLUS analysis, to be discussed in more detail below, are designated on the MDS solution by closed contours. Both the MDS solution and kindergartners' explanations indicate an emphasis on distance in the counting sequence of numbers as the basis for judgments of similarity. All kindergartners referred to counting at least once to justify their judgments, indicating that counting is a ubiquitous basis for numerical similarity judgments at that age level. The roughly horseshoe-shaped final configuration obtained for the kindergarten data

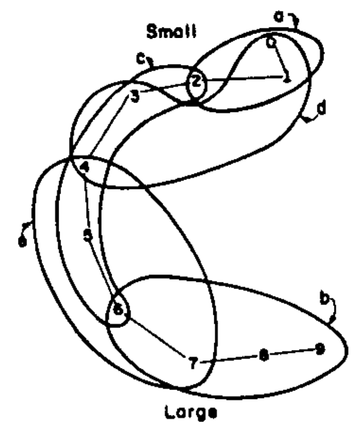

FIG. 1.—Kindergarten number-similarity judgments: final KYST-2A configuration (stress = .065). Curve connecting numbers indicates magnitude. Closed contours show clusters from the INDCLUS solution presented in Table 3. The two clusters least heavily weighted by kindergartners, *f* ("2,4,8") and *g* ("3,5,7,9"), were omitted from the figure.

TABLE 2

EXPLANATIONS GIVEN FOR SIMILARITY JUDGMENTS

Category	Example	Kindergarten (Numbers)	Third Grade (Numbers)	Sixth Grade (Numbers)	Kindergarten (Letters)
Counting distance	Picking "3,4" as most similar from "1,3,4" because "3 comes right before 4."	12	10	10	12[a]
Addition	Picking "1,4" as most similar from "1,4,8" because "4 is only 3 more than 1."	0	8	5	0
Even/odd	Picking "0,2" as most similar from "0,2,7" because "they're both even."	0	1	10	0
Multiplication	Picking "4,8" as most similar from "1,4,8" because "4 × 2 = 8."	0	0	10	0
Other magnitude	Picking "7,9" as most similar from "6,7,9" because "7 and 9 are both big."	2	2	2	1[b]
Other or no reason	Picking "2,5" as most similar from "2,5,7" because "they both go to 10 a lot, like 2 × 5 = 10."	3	1	1	2

NOTE.—Number of children in each group (out of 12) giving a particular justification at least once.
[a] Example: Picking "A,B" as most similar from "A,B,F" "because it goes A,B."
[b] Example: Picking "A,C" as most similar from "A,C,F" "because A is in the beginning and C is, too."

frequently indicates an underlying one-dimensional representation (Shepard, 1974), as is shown on the magnitude curve drawn on Figure 1. The disruption in this otherwise regular U-shaped configuration caused by the position of "0" is consistent with the confusion that some of the kindergarten subjects appeared to have over the proper place for zero. These results indicate that 5-year-old children are capable of making judgments about numbers that are presented to them as abstract symbols that correspond to an important numerical property (magnitude, or position in the counting sequence). This finding suggests that the numerical competence of older preschool children includes a stable representation of numerical relations that includes the feature of magnitude.

The MDS solution for third-graders' judgments is presented in Figure 2. Results for third graders generally resembled the kindergarten data, although "0" was clearly located in its proper place in the numerical sequence, in contrast to the confusion some kindergartners had concerning its proper relation to the other integers. A classification of the explanations given by third graders is presented in Table 2 and indicates increasing references to addition in contrast to the emphasis on counting by the kindergartners. The third-grade data are most notable for their similarities with the kindergarten results, suggesting that a similar type of representation of number (at least as assessed by this task) would be consistent with both counting and integer addition. This parallel between judgments explained with reference to counting and those explained in terms of addition is interesting in light of research (Groen and Parkman, 1972) suggesting that addition in early elementary school children consists of a rapid counting process.

Figure 3 contains the MDS solution for the sixth graders, which is notable for the substantial disruption of the systematic magnitude relation manifest in the kindergarten and the third-grade data. Although an axis has been drawn on which the integers can roughly project according to magnitude, it is evident that other criteria were also used by the sixth graders as they made their judgments. Sixth graders generally referred to oddness and evenness of stimuli and other multiplicative relations as a basis for their judgments, as can be seen from Table 2. The sixth-grade results begin to approximate the richness of the relations shown by the adult data from the Shepard et al. (1975) study.

Analysis of the adult data is presented in Figure 4. The adult solution quite closely

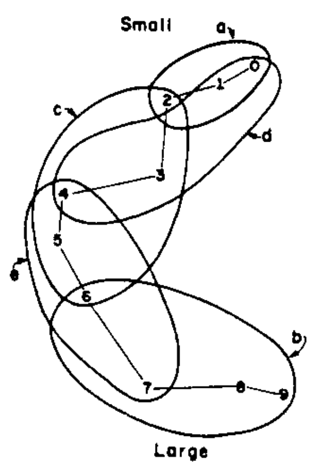

FIG. 2.—Third-grade number-similarity judgments: final KYST-2A configuration (stress = .070). Curve connecting numbers indicates magnitude. Closed contours show clusters from the INDCLUS solution presented in Table 3. The two clusters least heavily weighted by third graders, *f* ("2,4,8") and *g* ("3,5,7,9"), were omitted from the figure.

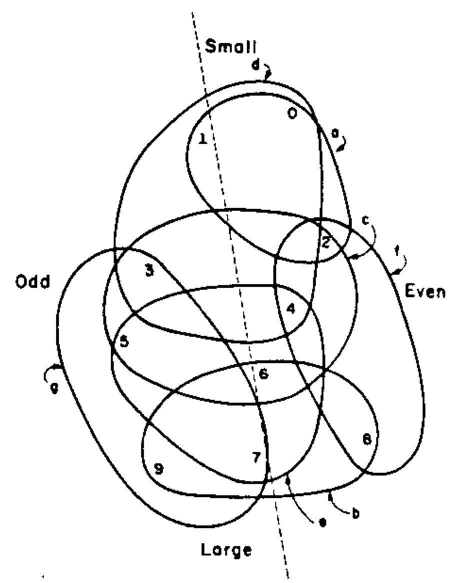

FIG. 3.—Sixth-grade number-similarity judgments: final KYST-2A configuration (stress = .133). Magnitude and Odd/Even dimensions are labeled. Closed contours show clusters from the INDCLUS solution presented in Table 3.

approximates that reported by Shepard et al. (1975) for their adult subjects judging numbers as abstract concepts. This solution also resembles that obtained for the sixth graders and provides a clear division into odd versus even numbers as well as a continuing sensitivity to magnitude. Analysis of judgments on a larger set of numbers would be necessary to provide firm support for the role that these multiplicative features appear to play in the judgments of adults and possibly of sixth graders. Use of larger numbers as stimuli would, however, preclude the comparison with younger children that the current approach permitted.

A listing of clusters obtained in the INDCLUS analysis of number-similarity judgments is presented in Table 3. Preliminary analysis suggested that seven-cluster solutions would provide a reasonable amount of variation in the kinds of clusters generated (magnitude-based clusters as well as some apparently reflecting multiplicative relations) while also accounting for a substantial proportion of the variance in subjects' similarity judgments. A series of seven-cluster solutions were then obtained, with the clustering solution presented in Table 3 representing the optimal solution in terms of interpretability and overall variance accounted for.

Clusters in Table 3 are ranked in terms of their relative weight in the kindergarten data. For each cluster listed in Table 3 the numerical weights are presented by age level, followed (in parentheses) by the rank of that weight relative to the other clusters at that age level. The first five clusters (*a,b,c,d,e*) seem to be generally magnitude based, consisting of series of numbers in order. An exception to this is cluster *d*, which consists of "0,1,3,4." This cluster was not readily interpretable and may reflect a limitation of the procedure of analyzing our data in terms of a set of common clusters.

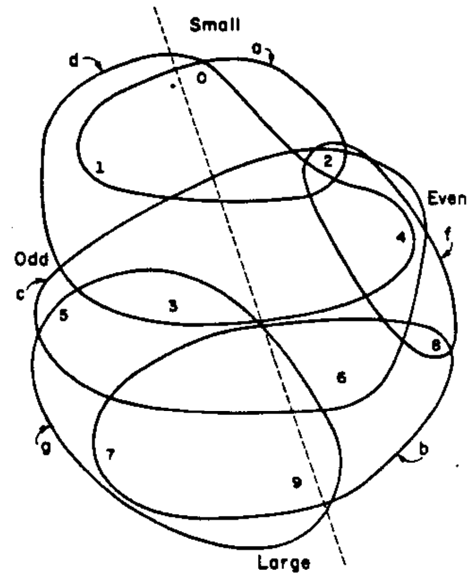

FIG. 4.—Adult number-similarity judgments: final KYST-2A configuration (stress = .142). Magnitude and Odd/Even dimensions are labeled. Closed contours show clusters from the INDCLUS solution presented in Table 3. The cluster least heavily weighted by adults, *e* ("4,5,6,7"), was omitted from the figure.

TABLE 3

INDCLUS SOLUTION FOR NUMBER-SIMILARITY JUDGMENTS: WEIGHTS FOR DIFFERENT SUBSETS BY AGE GROUP

Subset	Kindergarten	Third Grade	Sixth Grade	Adult	Elements of Subset	Interpretation
a	.585 (1)	.609 (1)	.451 (1)	.391 (2)	0,1,2	Small numbers
b	.477 (2)	.479 (2)	.374 (2)	.315 (3)	6,7,8,9	Large numbers
c	.343 (3)	.380 (3)	.303 (4)	.184 (5)	2,3,4,5,6	Middle numbers
d	.298 (4)	.319 (4)	.187 (6)	.163 (6)	0,1,3,4	Small numbers, excluding 2
e	.267 (5)	.288 (5)	.155 (7)	.070 (7)	4,5,6,7	Moderately large numbers
f	.135 (6)	.017 (7)	.337 (3)	.493 (1)	2,4,8	Powers of 2
g	.020 (7)	.064 (6)	.228 (5)	.283 (4)	3,5,7,9	Odd numbers, excluding 1
Additive constant	.234	.210	.242	.269	...	...
Variance accounted for (VAF) (%)	80.4	82.2	82.0	64.0	...	78.0 (Total VAF)

NOTE.—Rank of weight is given in parentheses.

The high similarity of the pair "2,8" (as part of cluster *f*) in the sixth-grade and adult data may have served to exclude it from a cluster that otherwise appears to consist of small numbers. The constraint of a common clustering solution across all of our subjects seems to have resulted in a "compromise" cluster without obvious interpretation, yet this procedure also permits evaluation of changes with development in the weight placed on particular clusters in a common solution.

While the first five clusters were the five most heavily weighted clusters for the kindergartners and third graders, the sixth graders and adults placed increasing weight on the last two clusters (*f*,*g*), which consist of the powers of two ("2,4,8") and the odd numbers ("3,5,7,9") excluding "1." For the adults, cluster *f* ("2,4,8") was the most heavily weighted cluster, preceding all magnitude-based clusters. Overall, the INDCLUS analysis suggests a developmental shift from exclusive reliance on magnitude-based relations toward increasing sensitivity to numerical features such as the odd/even relation and powers of two.

The above findings suggest that the acquisition of new applications for numbers and new numerical operations is accompanied by the inclusion of increasingly more complex numerical relations in the concept of "number." Analysis of the kindergarten data suggests that these judgments were based almost exclusively on the counting distance between numbers. Could this ability to judge similarity based on distance be extended to any well-learned ordered list, or does it reflect some special sensitivity to counting distance between numbers? A second study was undertaken to explore this issue. Two main modifications were made in the second study. Instead of the numbers 0–9, the first ten capital letters (A–J) were used as stimuli, and instead of presenting the child with the task of choosing which two numbers are "most closely related," children were given a more explicit criterion on which to base their judgments. Subjects were asked to determine from each triad which two letters are closest together and which two farthest apart in the alphabet song used to teach children to recite the letters of the alphabet in sequence.

Experiment 2

Method

Apparatus.—A letter wheel and apparatus identical in construction to that employed in the first study were constructed for the second study. Stimuli were made from the set of press-on dry transfer letters corresponding to the numbers used in the first study.

Subjects.—Subjects were 12 kindergartners, six boys and six girls (mean age 6-0), drawn from a private Philadelphia school. None of the subjects had participated in the first experiment.

Procedure.—Subjects were asked if they knew the alphabet song, and were requested to sing it. All subjects were able to sing the alphabet song without error. The procedure was otherwise identical to that employed in the first experiment, with the exceptions noted above.

Results

Consistency of judgments between repeated stimuli was comparable to that obtained in the first experiment, with reliability at 88% for judgments of those repeated triads. Proximity matrices were computed as in the first experiment, and concordance of these data with those obtained in Experiment 1 is presented in Table 1. These letter-similarity judgments are significantly related to the number-similarity judgments of kindergarten, third- and sixth-grade subjects, suggesting that position within a well-known ordered series is a significant factor in number-similarity judgments for all of our groups of children.

Results of the MDS analysis of the kindergarten letter-similarity judgments are presented in Figure 5. This analysis yielded a roughly horseshoe-shaped configuration similar to that obtained for the kindergarten

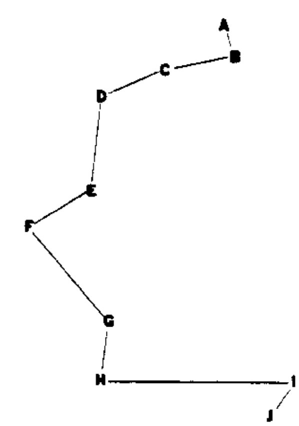

FIG. 5.—Kindergarten letter-similarity judgments: final KYST-2A configuration (stress = .090). Curve connecting letters indicates sequence in the alphabet.

number-similarity judgments (see Fig. 1). The distortions from a regular horseshoe shape contained in the KYST-2A output may indicate a tendency on the part of some of the kindergartners to show some local skewing of the alphabet, with accurate representation of the relations between, for example, G and H, but less understanding of the relation between these letters and the remainder of the alphabet. Alternatively, these small violations from a smooth horseshoe-shaped configuration may reflect chance fluctuations resulting from lesser familiarity with letters than with numbers on the part of kindergartners.

General Discussion

These two studies demonstrate three major points concerning the development of children's conceptions of number. The first concerns the ability of 5-year-old children to make consistent judgments about numbers presented as numerals reflecting a major numerical property (magnitude). Judgments by kindergartners about relations between numbers when given no context resemble closely their judgments about distance between the members of another familiar ordered list, the alphabet, when given an explicit criterion on which to make their judgments. Previous research by Gelman (1978; Gelman & Gallistel, 1978) has demonstrated that preschool children possess an extensive, principled understanding of numerical relations based on counting. The present results suggest that by the time children enter school they have extended this knowledge into a consistent understanding of the magnitude relations implicit in numerical symbols.

A second point concerns the demonstration of gradual expansion of the concept of number and the set of numerical relations on which similarity judgments are based over the course of elementary school, and apparently continuing until adulthood. This indication that there is a gradual expansion of the concept of number during the course of elementary school complicates Piaget's characterization of number development as the acquisition in middle childhood of a "number concept" based on one-to-one correspondence. In terms of the present study, it may make more sense to speak of a multiplicity of "number concepts," with development consisting of the mastering of a gradually increasing set of such concepts as children become sensitive to new numerical relations. It is important to note that we are not asserting that children necessarily are unaware of numerical relations that did not emerge from analysis of their proximity judgments. It is likely, for example, that at least some of our third-grade subjects know what odd numbers are, but they do not seem to view this as a basic characteristic of number to be used in judging similarity between numbers. This distinction between implicit and explicit numerical knowledge has been emphasized by Gelman (1982) as a key dimension along which preschoolers' understanding of mathematics differs from that of older children. In these terms, the present study investigated children's explicit knowledge about numbers, along with their willingness to include various numerical features in judgments of similarity between numbers. A gradual expansion with development of the set of features seen as determining this similarity was observed, which can be described as the overlaying of new relations on an early understanding of numerical magnitude.

A final conclusion from the present study is the relation between judgments of number similarity and numerical operations observed at all points in development. These numerical applications of counting, adding, and multiplying appear to have a profound effect on the process of expanding children's conceptions of what numbers are. Work with young children (e.g., Gelman & Gallistel, 1978) has demonstrated that much of preschoolers' conceptual knowledge about numbers is manifested in procedures such as counting, and that these procedures are guided by tacit numerical principles. In the same manner in which preschoolers' knowledge of counting plays a major role in the way they think about number, acquisition of additional numerical procedures seems to play a continuing role in determining how children and adults represent number. Our findings suggest that the mundane arithmetical skills and procedures that Piaget (1965, p. 29) dismissed as "merely verbal knowledge" may need to be given a more central place in determining the child's conception of number.

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