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Over the last decade cognitive psychologists have become increasingly interested in studying learning and performance in specific domains of knowledge. Among the domains that have received a great deal of attention are mathematics and science. Most of the research on mathematical and scientific thinking has been concerned with uncovering knowledge structures and reasoning processes in people of different levels of competence. How these structures and processes are acquired has only recently become a major concern. In this overview paper we will review some of the major recent research on mathematical and scientific thinking, giving particular attention to work that sheds light on the processes of learning and development. We will consider research in three areas of investigation: (1) the role of organizing schemata or structures in scientific and mathematical thinking; (2) the spontaneous construction or application of theories; (3) implicit understanding expressed as the invention of procedures.

THE ROLE OF ORGANIZING SCHEMATA IN THINKING

Researchers in the fields of cognitive development and cognitive psychology have concluded that problem solving in mathematics and science depends heavily on the kind of schemata, or structures, that children and adults bring to a problem. Although there have been challenges to Piaget's description of the structures available to children of different ages (see Gelman and Baillargeon, 1983, for a recent review), the Piagetian notion that individuals' cognitive structures determine the nature and power of their problem-solving abilities has a close relative in the schema [1] theory developed by cognitive scientists. In both accounts the available structures, or schemata, limit the range of problems one can successfully tackle. Further, which aspects of a problem are attended to, what interpretations they receive, and how they are approached are heavily

1. In current cognitive science usage, a schema is a kind of prototype of a situation. It describes the relationships that will be true across a number of specific situations. To take a simple example, a schema for a restaurant visit specifies that there is a room with tables at which one sits, that one gives an order to a waiter or waitress, that food is brought to the table, that at the end one pays for the meal, and so on.

dependent on the interpretive structures or schemata brought to the problem. Finally, it is recognized that these schemata are active rather than passive. They do not wait around to be brought into play by some input or sequence of experiences. Rather, they organize and direct behavior that sometimes even supplies the kind of inputs these structures require for their further elaboration.

Some of the most striking evidence for the conclusion that available schemata determine the course of problem solving comes from work on the difference between expert and novice problem solvers. It reveals that experts have different schemata available to them, and hence reason differently. Consider studies of physics knowledge.

In these studies, good but beginning students have been compared with advanced students or teachers. The studies show first that one's initial understanding, even of a simple textbook problem, depends upon one's level of knowledge in the field. Chi, Feltovich, and Glaser (1981) asked novices and experts to sort physics textbook problems on any basis they wished. Novices did so on the basis of the kind of apparatus involved (lever, inclined plane, balance beam, and the like), or the visual features of the diagram accompanying the problem. Experts classified the same problems on the basis of the underlying physics principle that was needed to solve the problem (e.g., energy laws, Newton's second law). Figure 1 shows some typical novice classifications; Figure 2 shows the contrasting expert classifications. Clearly, novices are more at the mercy of the way the problem is presented, while experts bring their own knowledge of important principles to bear in a way that reshapes the problem, usually in a more solvable form. This is much like the way in which good readers use their past knowledge about the topic or the form of discourse to impose a useful structure on a text, while beginning readers are much more victimized by poorly written material or indirect forms of expresssion (see A. Brown's paper in this collection).

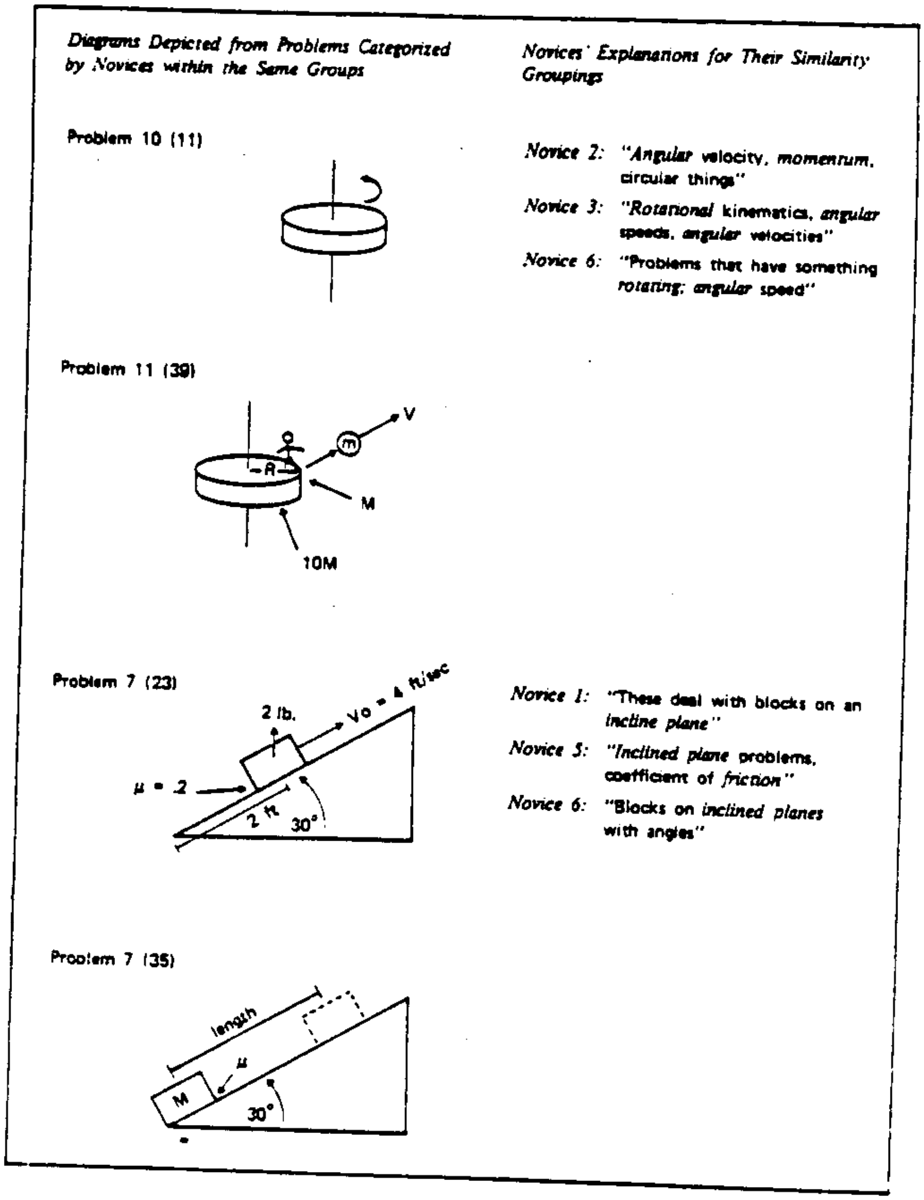

Figure 1. Diagrams depicted from two pairs of problems categorized by novices as similar and samples of three novices' explanations for their similarity. Problem numbers given are the chapter number and problem number from Halliday and Resnick (Physics, Edition. New York: John Wiley, 1974). (From Chi, Feltovich, and Glaser, 1981.)

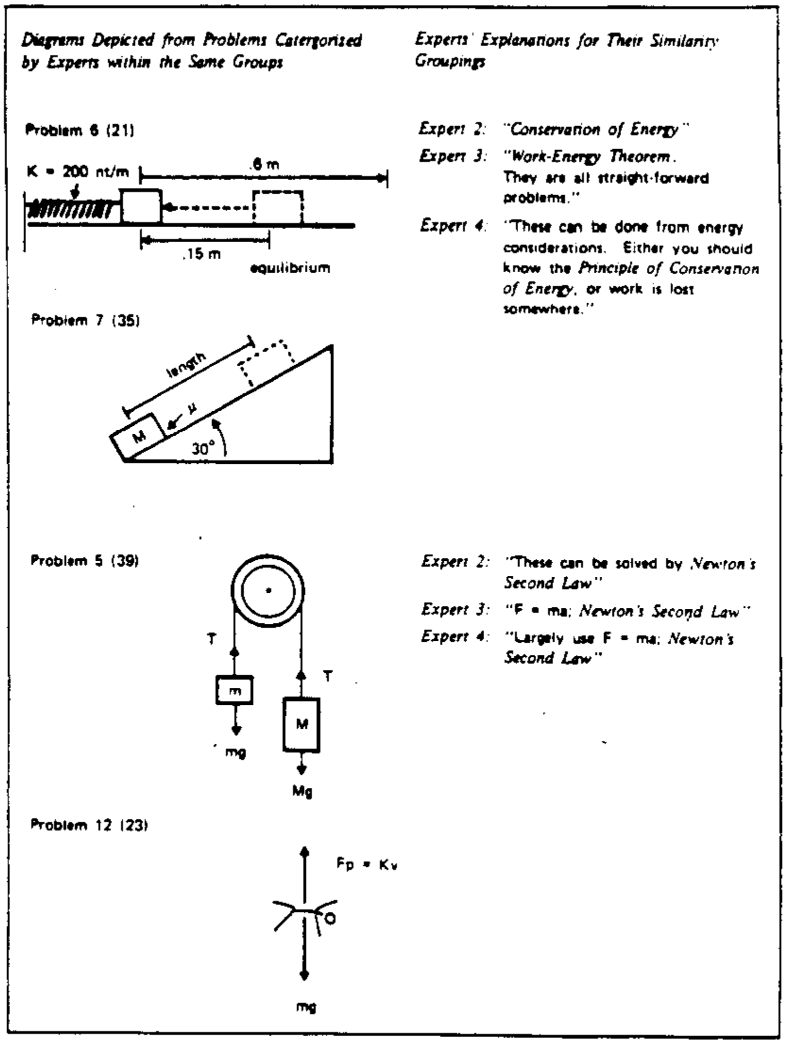

Figure 2. Diagrams depicted from pairs of problems categorized by experts as similar and samples of three experts' explanations for their similarity. Problem numbers given are the chapter number and problem number from Halliday and Resnick (1974). (From Chi, Feltovich, and Glaser, 1981.)

Initial differences between experts and novices in sorting and classifying problems are only the beginning, however. The process of solution is also different. What novices usually do is translate the given information directly into formulas. They then work on the formulas using rules of algebra, and usually they eventually come up with the right answer. Experts, by contrast, do not begin by translating into formulas. Instead, they work for awhile on reinterpreting the problem and specifying the various objects and relationships in the situation described. They may draw diagrams to express these relationships. By the time they are ready to write equations, experts have virtually solved the problem. They do much less calculation than novices—at least on the simple problems studied so far in this research. Experts, in other words, construct a new version of the problem for themselves, one that accords with the information actually given, but that is reformulated in terms of general principles and laws that make the solutions apparent (cf. Larkin et al., 1980).

Studies of people's knowledge of physics provide some of the most compelling demonstrations of the way differences in the kinds of schemata that are available affect problem solving. But these differences occur in other domains as well. Similar differences have been found in tasks as divergent as interpreting x-ray photographs and solving arithmetic problems. In each case, the more advanced problem solver does not simply respond to the problem in the terms presented, but instead reinterprets it in ways that reveal an underlying structure that makes the solution sometimes appear self-evident to the problem solver. This characteristic of "expert" problem solving can even be seen in the performances of very young children on arithmetic story problems. To illustrate we draw on the work of Riley, Greeno, and Heller (1983), which analyzes the converging data bases collected by Carpenter and Moser (1982), Nesher (1982), and Vergnaud (1982) in three different countries.

Riley and colleagues have identified three main classes of addition and subtraction word problems:

1. Those that involve a *change* schema—that is, situations in which an initial quantity is modified by virtue of its gaining or losing some amount. Example: David has 15 marbles. He loses 6 in a game. How many marbles does he have left?

2. Those that evoke a *combine* schema—that is, ones that describe the combination of subsets into a superset, or the decomposition of a superset into subsets. Example: In the class there are 35 children. Nineteen are boys. How many are girls?

3. Those that evoke a *comparison* schema. Example: Jack's group worked hard and planted 12 trees. Donald's group was slower and planted only 8 trees. How many more trees did Jack's group plant?

Among the most difficult problems for children are those that involve the comparison schema, and those that involve the change schema with the starting set unknown—for example: "Peter went out to play marbles with his friends. He lost five marbles in the game and came home with only eleven marbles in his pocket. How many did he have when he started out to play?" Up to the age of eight or nine, children have a great deal of difficulty with these problems, and they make characteristic errors. However, individual interviews with children show that once they master these problems, the answers become self-evident to the children. For the Peter-and-his-marbles problem they say things like, "11 plus 5 is 16, so he had 16 when he started," without being able to tell the interviewer how they knew that they should add the two numbers. This is especially striking when we consider that the story describes *losing*, and this would most naturally prompt children to want to subtract instead of add (indeed that is what many children who fail this kind of problem do).

How do these "expert" eight- and nine-year-old problem solvers arrive at the idea that they need to add to solve a problem in which a child loses a number of things? That is the kind of question that was addressed in two recent efforts to build formal theories of the knowledge structures and reasoning processes used by children when they solve story problems. Riley, Greeno, and Heller (1983) developed a family of computer simulation models that solve problems from each of the three classes at three different levels of competence. Only the most competent model is able to solve problems like those about Peter and his marbles. To do so, it first classifies the problem as a change problem and then calls upon its change schema to interpret the situation—much as the expert physicist sorts problems according to the kind of physics laws they invoke. Then, in order to reason about a starting set of unknown size, it reinterprets the problem in terms of a part-whole schema. In this reinterpretation, it recognizes that the whole is made up of two subsets of marbles, the five that Peter had at the beginning but lost and the eleven that he had at the beginning and kept. Because the part-whole schema specifies that parts can be combined to make up a whole, the system "knows" that it should add eleven and five.

An alternative story-problem theory developed by Briars and Larkin (1981) solves these problems not by calling on a change schema but by constructing a mental script that reflects real-world knowledge about separating and combining objects rather than using the more abstract schema proposed by Riley, Greeno, and Heller. The script describes the actions in the story and allows the system to keep track of the sets and subsets involved. Yet, in Briars and Larkin's model, too, it proves possible to solve a difficult problem such as Peter and his marbles only by calling on a part-whole schema.

THE SPONTANEOUS CONSTRUCTION AND APPLICATION OF THEORIES

Another recurrent finding in mathematics and science learning is that people regularly construct theories for themselves. One line of evidence for this is that the beliefs they hold about how the physical world works or about the properties of numbers are not simple reproductions of what they may have been taught. Studies of physics learning highlight the fact that people bring with them to their school or university science courses a tenacious set of "spontaneous theories" about how the physical world works (see, for example, Champagne, Klopfer, and Gunstone, 1981; McCloskey, 1983; Selman et al., 1981). These theories are often fundamentally inconsistent with the modern scientific theories that are to be taught, but they are robust and are not readily abandoned as the result of instruction. There is evidence that students adopt the school-taught theories for solving textbook problems, but resort to their prior spontaneous theories when asked to solve problems that are different from those drilled in class. An example comes from McCloskey's studies of university students' responses to questions about the path of a moving object as it emerges from a circular tube. They typically answer that the object will continue to move around in a circle, just as it did in the tube! Such answers are more consistent with the medieval impetus theory of moving inanimate objects than with that of Newton.

Three lines of evidence support the view that spontaneous or naive theories are constructed in the domain of mathematical knowledge: (1) Children invent procedures they could not have invented unless they had constructed theories; (2) In some cases children systematically "err" in a way that can be traced to a misapplication of their theory; and (3) In some cases children work through the implications of the knowledge they already possess.

Simple Addition and Subtraction

Consider research on a problem in learning long thought to be a prime example of rote acquisition of associations: simple, single-digit addition and subtraction problems. School textbooks typically define addition as a process of counting objects to represent each addend, combining the subsets thus created into a single large set, and then recounting the combined set. Teachers generally expect children rather quickly to have memorized the answers to simple addition problems (that is, to have learned the sums table) and thus to cease to depend upon any form of counting. Research in several countries, however, has now made it clear that there is a period of time in which children continue to use a counting method to do addition. Further, they use a different procedure from the one they were taught. Most children use a procedure that is more elegant than the one they were taught, because it minimizes the computational steps and because it appears to involve an intuitive appreciation of the mathematical principle of commutativity. What children typically do is behave as if they had a counter in their heads. They initially set this counter to the larger of the two addends, and then increment it by a number of steps equivalent to the smaller. For example, to add 3+5, the child starts at 5 (even though it is named second) and counts on: "5.....6, 7, 8." The final count ("8") is then given as the answer. This procedure has been documented in reaction-time and interview studies of a number of children in different countries and of different measured mental abilities (Groen and Parkman, 1972; Svenson, 1975; Svenson and Broquist, 1975). A study by Groen and Resnick (1977) shows that the procedure can be invented by children as young as four or five years as a result of practice in addition—with no direct instruction, demonstration, or explanation.

A similar story can be told for subtraction. Typically, textbooks demonstrate either of two procedures: a counting-out procedure in which a starting set (the minuend) is established, a specified number of objects (the subtrahend) is removed, and the remainder counted; or a matching procedure in which sets to represent two quantities are established, objects from these sets are paired one-for-one, and the remaining unmatched objects are counted. However, after practice, children do something rather different from either of these: they either count down from the minuend or count up from the subtrahend, <u>whichever will take the fewest counts</u>. Thus, for the problem 9 - 2 they say, "9...8, 7" and answer "seven" and for 9 - 7 they say, "7...8, 9" and answer "two" (Wood, Resnick, and Groen, 1975; Svenson and Hedenborg, 1979). It is as if the children who invented this procedure understood the complementarity of addition and subtraction. Furthermore, it is not just a shortcut—a cropping of redundant steps in the algorithm that had been taught—for it involves, for each case, a decision whether to count down or up. It involves a true invention of a new procedure.

Studies of the above kinds demonstrate the centrality of invention even in apparently simple and "rote" domains of learning. However, they should not be taken to imply that inventions are always successful. Systematic "errors" are well documented. Where they were once attributed to carelessness or lack of any systematic understanding of a system, they are now recognized as being based on erroneous conceptions of how the system works. In the case of arithmetic, it has been shown that systematically used wrong procedures are variants of the correct ones. They are analogous to computer algorithms with "bugs" in them, and thus have been labeled "buggy algorithms." A finite number of bugs, which in various combinations make up several dozen buggy algorithms, have been identified for subtraction—which is the most intensively studied arithmetic domain so far. The children who display these buggy algorithms are systematically applying rules that no one could have taught them (for presumably no one would deliberately teach them a wrong rule). Buggy algorithms are thus clear examples of inventions that are unsuccessful.

Despite their failure as rules of calculation, buggy algorithms demonstrate an important characteristic of human learning and performance. From close analysis it is clear that most of the various incorrect algorithms that have been observed among children are small and often quite sensible departures from the correct algorithm. As the examples in Figure 3 reveal, buggy algorithms tend to "look right" and to obey a large number of the important rules for written calculation: the digit structure is respected, there is only a single digit per column, all the columns are filled, there are crossed out and rewritten digits, and so forth. Each buggy algorithm looks like an orderly and reasonable response to a new situation, although each violates a fundamental rule of the arithmetic system: the necessity of maintaining the value of the top quantity whatever particular transformations or exchanges of quantities may be made between the columns in the written number.

Such buggy algorithms point to a pervasive feature of learning and cognitive performance: people seem to try to make sense out of the world, and to create rules for acting in it, even given limited data; they do not wait until all the information is in before they start to construct a "theory" to account for what they have before them. In the case of buggy subtraction algorithms, children seem to construct a "theory of allowable operations" that respects all the information they do have while ignoring a mathematically important constraint that is apparently not adequately stressed in primary school arithmetic teaching.

1. **Smaller-From-Larger.** The student subtracts the smaller digit in a column from the larger digit regardless of which one is on top.

   328 − 117 = 211   542 − 389 = 247

2. **Borrow-From-Zero.** When borrowing from a column whose top digit is 0, the student writes 9 but does not continue borrowing from the column to the left of the 0.

   602 − 437 = 265   802 − 396 = 506

3. **Borrow-Across-Zero.** When the student needs to borrow from a column whose top digit is 0, he skips that column and borrows from the next one. (Note: This bug must be combined with either bug 5 or bug 6.)

   802 − 327 = 225   604 − 458 = 309

4. **Stops-Borrow-At-Zero.** The student fails to decrement 0, although he adds 10 correctly to the top digit of the active column. (Note: This bug must be combined with either bug 5 or bug 6.)

   703 − 678 = 175   604 − 387 = 307

5. **0 − N = N.** Whenever there is 0 on top, the digit on the bottom is written as the answer.

   709 − 352 = 457   6008 − 327 = 6321

6. **0 − N = 0.** Whenever there is 0 on top, 0 is written as the answer.

   804 − 462 = 402   3050 − 621 = 3030

7. **N − 0 = 0.** Whenever there is 0 on the bottom, 0 is written as the answer.

   976 − 302 = 604   856 − 409 = 407

8. **Don't-Decrement-Zero.** When borrowing from a column in which the top digit is 0, the student rewrites the 0 as 10, but does not change the 10 to 9 when incrementing the active column.

   302 − 158 = 344   205 − 9 = 1106

9. **Zero-Instead-Of-Borrow.** The student writes 0 as the answer in any column in which the bottom digit is larger than the top.

   326 − 117 = 210   542 − 389 = 200

10. **Borrow-From-Bottom-Instead-Of-Zero.** If the top digit in the column being borrowed from is 0, the student borrows from the bottom digit instead. (Note: This bug must be combined with either bug 5 or bug 6.)

   702 − 358 = 454   508 − 439 = 109

Figure 3. Samples of buggy subtraction algorithms invented by children. (Adapted from Brown and Burton, 1978.)

A further consideration of the origins of buggy arithmetic algorithms highlights this point. Brown and VanLehn (1982) have developed a computer simulation program that invents the same subtraction bugs and therefore makes substantially the same errors children do. This program serves as a formal theory of what children might be doing when they invent buggy algorithms. According to this theory, buggy algorithms arise when the procedures the child has previously learned are incomplete. The child, trying to respond, eventually reaches an impasse, a situation for which no action is available. At this point, the child tries to fix her procedure, calling on a list of "repairs"—actions to try when the standard action cannot be used. The repair list includes strategies such as performing the action in a different column, or substituting an operation (such as incrementing for decrementing). The outcome generated through this repair process is then checked by a set of "critics" that inspect the resulting solution for conformity to some basic criteria, such as no empty columns, only one digit per column, only one decrement per column, and the like.

Together the repair and critic lists constitute the key elements in a "generate and test" problem-solving routine. This is the same kind of "intelligent" problem-solving that characterizes many successful performances in other domains (see Simon, 1976). With buggy algorithms, the trouble seems to lie not in the reasoning processes but in the inadequate data base applied. In particular, the critic lists do not contain criteria that would reject repairs that violate the principle of maintaining quantity equivalence. The invented algorithm is a sensible construction, but on a data base that is incomplete. It therefore turns out to be a "buggy" rather than a successful invention.

Repair theory is, in fact, a detailed theory of acquisiton for a small domain of arithmetic. Its broader implication is that cognitive theories of acquisition must recognize people's tendency to organize and structure whatever information they have—even though the information may be grossly incomplete or downright inaccurate. People do not simply acquire information passively until there is enough of it for correct rules and explanations to emerge. Instead they construct explanations and rules of procedure continuously. This tendency to construct ordered explanations and routines can account at least partly for the phenomenon, discussed above in the context of physics learning, of spontaneous theories that are resistant to change even when instruction (and thus better information) does come along. The naive theories have been constructed to help the individual make sense of the natural world. Like buggy algorithms, they are partly correct. To give them up in order to accommodate the principles of Newtonian mechanics is to give up a long-held system of knowledge, with many interrelated schemata and domains of application, for a new theory that is "incoherent" (because unconnected either to other schemata or to practical experience). It is not surprising—although it is disturbing—that many students find it easier to simply reserve their classroom-acquired theories for classroom situations and do not try to apply them outside.

Measurement by Very Young Children

Preschool children have a very limited understanding of measurement, but as Miller (1982) shows, this does not stop them from negotiating measurement tasks. What they do provides support for the idea that they, too, use an available theory, in this case one about counting, and do their best. That is, they spontaneously misapply their implicit theories about numbers when confronted with the task of measuring continuous quantity. An example comes from Miller's observations of what three-year-old children often do when they are asked to give two friends the same amount of water to drink. They seem to think that this is accomplished by equating the number of times they pour (independent of how long or how much they pour) into each friend's glass. They fail to realize that the measurement of continuous quantity requires that the units to be counted be equal (see Gelman and Baillargeon, 1983, for a discussion of similar tendencies in somewhat older children). No such problem arises when discrete objects are simply counted: indeed, as Gelman and Gallistel (1978) point out, any collection of disparate objects can be counted. In the counting of discrete objects, it does not matter whether the items are the same size, color, shape, type, and so on. In contrast, it does matter what units are counted in the act of measuring—they must be equal.

It is for this reason that one can say that the young children in Miller's experiments misapplied their theory of counting to the task of measurement. Still, although young children may have a limited understanding of measurement and the requisite need for equal units, it is nevertheless clear that they, too, invent solutions to the problem at hand—solutions that depend on an already available theory in a related domain.

Infinity

For our final example demonstrating that children work up theories in the domain of mathematics, we consider work on the development of a beginning understanding of infinity (Evans, 1983; Gelman and Evans, 1981). Children between the ages of five and nine participated in interview studies designed to assess their level of understanding that one may always add "one" to a positive integer and get another number and thus that there is no largest number. The children were first asked what was the biggest number they could think of (or what they could count to). Then they were asked what would happen if one were added to their designated number, one more to that number, one more again, and so on. Interspersed with the questions about the effects of repeated addition were questions about whether anyone could find a larger number than the child had, if anyone could count past the place at which the child insisted on stopping (if he did), whether there is a largest number, and the like.

Children's understanding of infinity was related to their ideas of what constituted a "big" number and how well they could count. To illustrate, many of the youngest children could count only to 20 or 30. The numbers they mentioned as "the biggest" were usually less than 100 or else made-up numbers like "twenty-eight-thirty-two." These children claimed they could not add one to the number they said was the biggest they could think of. Children classified at the second level of understanding of infinity typically mentioned very large numbers, such as a million, in response to the question about the largest number they could think of. They generated large numbers in an organized fashion, even when their answer was incorrect, such as that one million plus one is two million. Still, they often said that there is a largest number to which nothing may be added—whether or not they said they did not know what it was and/or that no one ever could know. Some children at this level even said one could keep adding and always get yet larger numbers, but paradoxically, insisted that there must be a largest number. Thus, even though these children recognized the effect of continued iteration on the size of a number, they failed to recognize its consequences. Finally, the most advanced children were not only able to give very large numbers like a million at the start. They said that one can keep adding and thereby generating yet larger numbers, and that the count numbers are unbounded, that is, that there is no largest number.

How do such findings support the notion that children construct theories about numbers? First, they illustrate that their concepts and abilities are interrelated. Children who have limited counting abilities take rather smallish numbers to be the largest they or anyone could know (although they must hear talk of much larger numbers), use their limited knowledge as evidence for there being a largest number, and hence deny that continued iteration will necessarily yield larger numbers. Siegler and Robinson's (1982) work with children of comparable ages suggests that these same children would also lack an understanding of the base-10 rule that underlies the English (and most other European) count-word sequence.

Second, they illustrate that children use what they know to make further conceptual progress. As Gelman and Evans (1981) note, some of the more advanced children came to the interview without having realized that there is no largest number. Although the interviewer did interact with the children, she never answered questions for them; she never said "there is no largest number." It appears that the interview established conditions under which the children were able to explore for themselves the implications of the questions asked. Children who initially said that there was a largest number came to realize that their belief in their ability always to add to any number they could think of implied that there was no largest number. The idea that some of the advanced children treated the experiment as an occasion to explore the implications and limits of their theories of number derives support from the fact that other children who knew the answers at the outset of the interview said they had done just this at an earlier time. They said they discovered that the numbers never end on their own, or in conversations they initiated at home when they found no matter how long they counted they never reached the end, or in conversations about numbers with their peers. This is evidence for the view that theories—or, more properly, pieces of theories—serve to motivate further theory development (Karmiloff-Smith and Inhelder, 1974/75).

Although children in the early years of elementary school did rather well in the infinity study, they lacked complete understanding of the concept(s) of infinity. Evans (1983) suggests that the children in these studies revealed knowledge most closely approximating the intuitive understanding of the early Greeks. It is not unreasonable to propose that more modern and formal concepts of infinity could prove as difficult to master as are the more modern theories of physics. Cantor's proofs regarding transfinite numbers bewilder many an undergraduate introduced to them by R.G. (just as they outraged many of Cantor's contemporaries). Clearly then some theories develop with relative ease and others with only considerable effort (for a discussion, see the paper by Susan Carey in this collection). This does not change the fact that theories, be they correct or not, are constructed by adults and children alike to make sense of matters scientific and mathematical.

THEORIES ARE OFTEN IMPLICIT IN PROCEDURES

Much of what people—especially young children—know is implicit in their procedures, rather than being something they are able to make explicit. The distinction between implicit and explicit knowledge is well known in psycholinguistics. Young children are granted implicit knowledge of linguistic structures well before they are granted explicit, or stateable knowledge of any of these (Gleitman, Gleitman, and Shipley, 1972). They are granted implicit knowledge because the sentences they speak can be shown to be rule governed. The strongest evidence for the latter inference comes from those sentences that young children utter that they could never have heard but that nevertheless can be traced to an implicit rule or structure. A similar distinction is necessary in the domain of mathematics knowledge.

When Gelman and Gallistel (1978) concluded that even young preschoolers know how to count, they characterized this knowledge with reference to five principles: (1) the one-one principle: each item in an array must be tagged with one and only one unique tag; (2) the stable-order principle: the tags used must be drawn from a stably ordered list; (3) the cardinal principle: the last tag used for a particular count represents the cardinal number of the array; (4) the abstraction principles: any set of items may be collected together for a count; and (5) the order-irrelevance principle: the order in which items in a set are tagged is irrelevant. Gelman (1982) notes that although the evidence points to the conclusion that preschoolers have implicit knowledge, it does not follow that they have explicit knowledge of the counting principles. Thus, there is no reason to presume that preschoolers can articulate the cardinal principle. Yet their behavior supports the conclusion that implicit knowledge of it is available. For example, when young children count large sets they often err and fail to indicate the cardinal value represented. However, when they watch a puppet count sets just as large and hear it answer an "how many is that" question erroneously, they nevertheless often can say that the puppet gave the wrong answer and then correct the puppet (Gelman and Meck, 1983).

Greeno, Riley and Gelman (1984) show that one can describe the counting principles with reference to a small set of action schemata. Referring to this account as the conceptual competence the child brings to a counting task, they develop a formal account of how the conceptual competence can be linked to the performance competence that children exhibit on a range of counting tasks. Performance competencies are granted when a child can assemble a set of procedures that will produce the required performance that adheres to the principles contained in the statement of conceptual competence. Such efforts make it possible to articulate the notion of implicit knowledge and hence circumvent the need to have people state their knowledge before granting them an understanding of principles. What follows provides support for the view that it is important to find ways of using children's procedural performances as indicators of the nature of their implicit understanding.

Understanding Implicit in Children's Invented Addition Procedures

The role of conceptual understanding implicit in invented procedures is revealed in work done by Robert Neches (1981; Resnick and Neches, 1983); this work attempts to provide a formal account of how children invent the addition procedure (described above) of counting on from the larger of the two addends. Neches has constructed two versions of a computer simulation program that begins with the procedure taught in school and modifies itself so that after a number of problems it performs the procedure of counting on from a larger number. These programs are of interest because they show how a few simple heuristics for examining and modifying procedures can cumulatively produce large changes in performance, without need for external intervention. They are also of interest because comparison of the two versions shows how mastery of a key (implicit) conceptual principle can permit learning of a new principle. In the first version the program (called HPM), can invent the count-up-from-larger procedure only if it already "knows" that pairs of problems with the same addends are particularly interesting. Such knowledge is needed because the system proceeds by noticing that these "commutative pairs" yield the same answer, and then applies a heuristic that selects the most efficient of two procedures when the two yield the same answer. Counting-on-from-larger is the procedure with the fewest counts needed, so it is selected.

Although this theory is plausible up to a point, it cannot account for the finding (Groen and Resnick, 1977) that children invent the count-on-from-larger procedure even when practice in addition has been deliberately arranged so that commutative pairs never appear in succession. HPM cannot invent under these conditions because the demands on working memory become excessive. However, a second version of HPM can solve this problem. In the second version, HPM is given at the outset a strong version of Gelman and Gallistel's fifth principle of counting: the principle of indifference to order—the concept that while number names must be assigned in a fixed order when counting, it does not matter which object receives which number name. In HPM's strong version of this principle, the system is totally indifferent to which objects are counted, and it treats as "the same quantity" any count of objects that arrives at the same ending number regardless of which objects have been counted. With this higher-level (but plausible for young children) understanding in its repertoire, HPM is able to apply its procedure-changing heuristics without an excessive demand on working memory. Although the new version of HPM cannot be said to "know about" commutativity in the sense of explaining it, it behaves "as if" it knew commutativity. It is a task for the next stage of work on this problem to show that regular performances of this kind can become the basis on which a learning system can construct new schematic (or conceptual) knowledge.

Even when children are taught procedures directly, they often must implicitly understand many underlying principles in order to successfully incorporate those procedures into their own conceptual competence. This is nicely demonstrated in a detailed case study of a child who took several months to learn a procedure for equalizing two sets, even though that procedure was directly demonstrated to him and required no arithemetic that was beyond his capabilities at the beginning of the learning period. The problems given to David were of the form, "You have four cookies and I have six. What can we do so we each have the same number?" At the beginning of the study David was able to solve these problems in two ways. He could find the difference between the two sets and have the person with the larger number give away that many ("You could sell two"), or have the person with the smaller number acquire that many more ("I could buy two"). He could also (in his head) combine the two sets and then give half the total to each person ("We could put them in the middle and each take five"). David could use these procedures interchangeably and with great flexibility; he could also apply them to quite large set sizes, demonstrating very good facility with mental arithmetic. However, he could not, even after it was demonstrated successfully, use a third procedure, one of direct transfer from one set to another ("You could give me one").

An analysis of the formal demands of the buy/sell and transfer procedures and of David's performances reveals what it was about transfer that gave David difficulty. In buy/sell, only one of the two sets is changed, and it is changed by exactly the difference between the two sets. The effect of any mental operation is local—that is, what is done to one set does not affect the other. In transfer, however, a change in one of the sets is linked to a change in the other. David had to coordinate these changes in order to use the transfer procedure. In addition, the number of objects to transfer is not directly derivable from the difference between the two sets; it is half of that difference.

To master transfer David had to understand these two aspects of the equalizing situation. His protocols, taken over a five-month period, show that his first step was to understand the double effect of a transfer. That is, for any two starting sets, if the experimeter suggested a number to transfer, he was able to state whether it would or would not equalize the sets and why (that is, he could say how many each person would have after the transfer). For several months, however, he could not decide himself how many to transfer; he did not even have a systematic trial-and-error procedure. David's first step toward solving the problem of how many to transfer was to construct a set of specific rules. About three months after the beginning of the study, he was able to state that any time the difference between the sets was two, a transfer of one would equalize them. He generated many examples of this, including a case where the initial sets were 158 and 160; and a case where the initial sets were 40 and 60 (which he said was a difference of 2 times 10, so one should transfer 1 times 10!). He had thus used his understanding of the dual effects of transfer to construct an empirical rule. However, he had not yet mastered the principle of "splitting the difference." Further study of David over the next two months showed that he eventually mastered this principle as well. However, he showed this understanding first as a procedure (he systematically found the difference between the two initial sets and then transferred half of that difference), but he could not

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2. This case study is being conducted by Terry R. Greene.

explain or describe what he was doing. Subsequently, he was able to describe his procedure, although at the end of our observation period he could not yet explain why it worked.

SUMMING UP

The research we have described above establishes the outlines of a set of questions on the nature of learning that have only begun to be addressed. First, we know that the kinds of schemata that people have make a difference in their methods and levels of success in problem solving. We know that experts in any domain have different schemata from those of novices in the same domain. Sometimes experts' schemata are elaborations and refinements of those of novices. Sometimes, however, experts' schemata are in conflict with novices'. In these cases, becoming an expert would require giving up or substantially restructuring one's original schema. These novice-expert contrasts pose a problem in the psychology of learning. Because all experts were once novices, we need to know what the processes are by which people construct new schemata, or modify existing ones.

Second, evidence that people tend to construct spontaneous theories further underlines the importance of this fundamental question. Spontaneous theories, because they are often "wrong" and it is thus certain that no one would have taught them, <u>have</u> to be constructed by learners themselves. What are the processes of such construction? Finally, the research on the relations between understanding and procedural knowledge helps to highlight our current lack of adequate theories of how people build and modify schemata. We are able to show quite clearly that conceptual (schematic) knowledge underlies procedural inventions. We also have a number of detailed models of how procedures are constructed by people (see Anderson, 1981, as well as the work by Neches, 1981, and by Greeno, Gelman, and Riley, 1984). But we do not yet have very <u>specific</u> models or theories of how the schematic knowledge itself is constructed. This is not likely to be a problem much longer, however. For many, ourselves included, have turned our attention to this very matter (see, for example, Sternberg, in press).

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