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TO KNOW MATHEMATICS IS TO GO BEYOND THINKING THAT "FRACTIONS AREN'T NUMBERS".

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Introduction

Our charge was to highlight matters of method, to emphasize our use of non-standardized tests to explore mathematical competence. Discussions of method are woven into the presentation of our work on the child's understanding, actually misunderstanding, of fractions. We wanted to uncover biases Kindergarten, Grade 1 and Grade 2 children might bring to their lessons on fractions and since there are no standarized fraction tests for this age range, we had to develop our own assessments.

Some Universal Findings

All over the world, both young children and non-schooled adults use counting algorithms to solve arithmetic problems. These are often made up and resemble those that schooled children invent. Whether schooled or unschooled, individuals have strong tendencies to decompose natural numbers into manageable or known components, to count when adding or subtracting, and to use repeated addition (or subtraction) to solve "multiplication" (or "division") problems. These multiplication and division solutions are used both before and after children have been taught more standard multiplication and division algorithms in school. They are also used by un-schooled adults and children in a variety of settings and cultures (e.g., Carraher, Carraher, & Schliemann, 1985; Ginsburg, 1977; Groen & Resnick, 1977; Lave, 1988; Resnick, 1986; Saxe, 1988; Saxe, Guberman & Gearhardt,1987; Starkey & Gelman, 1982)

Some of the algorithms invented by older children and un-schooled adults are more complex than those used by preschool children. Still, as Resnick (1986) notes in her analysis of these, all use principles of counting and addition (or subtraction) with the positive integers. An example from her own work illustrates this. Pitt (7 years-7

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mos) said that he solved "two times three" as follows:" *...two threes...one three is three, one more equals six*". Similarly, schooled and unschooled individuals in Africa and Latin America use a combination of number decomposition moves and repeated additions (or subtractions) to solve the multiplication problems presented by investigators. Young children are also less able to work with large numbers; otherwise, it is hard to distinguish their invented, out-of-school, solutions from older childrens' and adults'. (Carraher et al.,1985; Saxe,1988). Lave, Murtaugh, and de la Rocha's (1984) work with shoppers in a California supermarket provide an especially compelling documentation of how everyday "intuitive" solutions for determining unit prices are preferred over any taught in school.

Elsewhere we have suggested that the widespread invention of algorithms that are based on counting and/or repeated addition (subtraction) algorithms provides further support for our conclusion that a universal set of implicit principles governs the acquisition of initial mathematical concepts (Gelman,1982; Gelman & Meck, 1986). Our idea has been that a skeletal set of counting principles, in conjunction with some principles of addition and subtraction, promote the uptake of mathematical data that are relevant to these principles. The skeletal principles provide the a priori structure necessary for learners to notice, assimilate, and store relevant data in an organized manner (Gelman, in press; Gelman & Greeno, in press). For example, young children's implicit knowledge of the counting principles guides their initial attention to the counting sequence(s) in their language, the learning of which promotes the further development of these principles. It guides attention because the use-rules for the conventional counting string are consistent with the dictates of the counting principles. Since the use-rules for names of basic objects are not (Gelman & Meck,1986), children can avoid mixing up the function of labels and count words, even when both kinds of words are used for the same entities. Of course, once these different classes of words are sorted out, further learning is necessary. But it is now clear that this learning will occur whether or not children attend school.

Many have contrasted the discrepancy between the ease of learning counting and counting-based arithmetic solutions in everyday setting with the difficulty of learning school-taught mathematics (e.g. Lave, 1988; Resnick,1987; Saxe, 1988). What should schools do about this discrepancy? Should they make a concerted effort to adopt teaching models that more closely resemble the learning that occurs in everyday situations? This could be a reasonable policy if both the assumptions that implicitly underlie this recommendation hold. These are that schooling should build on what children already know or find relatively easy to learn, and that children bring to school a suitable foundation upon which to develop further mathematical knowledge.

We share the view that instruction should take into account the knowledge base a child brings to school. Nevertheless, we question whether this provides a ready base upon which to achieve the goals of mathematics teachers -- to teach mathematics. We are beginning to worry that learners' inclinations to apply their initial theory or intuitions about number to the novel inputs offered in schools interferes, for a very long period of time, with mastery of crucial material. Although their initial, pre-instructional assumptions about the nature of numbers serves many well in everyday interactions, these assumptions may contribute to the difficulty of learning mathematics taught in school.

Fractions Are Not What You Get When You Count Things

Although there are principles that both guide and structure early learning about counting, we think that this can be a mixed blessing. On the one hand, learning to count and add without such principles is exceedingly hard, even with the aid of highly structured input. For example, for Down's Syndrome children, the large majority of whom appear not to be guided by an intuitive understanding of counting, learning to count progresses very slowly or not at all, despite intensive drill (Gelman & Cohen,1988; Irwin,1989). This is all the more remarkable in that learning to count would appear to lend itself to rote-learning approaches.

On the other hand, there comes a point at which what must be learned transcends the principles that guide early learning. Much of mathematics involves operations and entities other than counting and the addition and subtraction of the counting numbers (or the positive integers). When it comes time to transcend the early knowledge built upon the counting principles, these principles may hinder progress almost as much as they promote it. Insofar as the principles that guide early arithmetic learning do not incorporate more advanced operations with numbers, and insofar as they do not work without modification when applied to numbers that are not count numbers, difficulty in learning modern mathematics might well be the rule as opposed to the exception. To illustrate why, we need not go too deeply into mathematics. The concept of a fraction will serve our purposes, especially since it seems to be a watershed in elementary school mathematics learning (e.g. Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980).

We believe that the data on unschooled mathematical abilities are consistent with the conclusion that our first notion of a number amounts to a belief that: "Numbers are what one gets when one counts things". If this characterization captures initial conceptions of number, then learning that fractions are numbers *should* be difficult. Fractions are numbers generated by the division of 2 numerosities, *they are not count numbers*. Thus, we should not presume that the requisite principles for dealing with entities like fractions exist when they are first introduced in school. To show why, we review briefly Gelman and Gallistel (1978).

Gelman and Gallistel distinguished between *numerons* and *numerlogs* to highlight the difference between the conventional linguistic counting tags (numerlogs) employed by a culture, and the mental entities (numerons) to which the numerlogs are mapped. Nowhere in their account of the implicit system of early mathematical abilities is there an entity that corresponds to the division of one numerosity by another. Nor could there be. Numerons are what one gets when one counts items. In general, the result of dividing one numerosity by another is not a countable numerosity and cannot therefore be represented by a numeron. At least on the basis of current data, there is no justification for attributing an arithmetic or mathematical understanding of division to

young children. They might know about cutting things into parts (Miller, 1984), but this is quite a different matter than knowing about dividing one numerical representation by another. Indeed, Kerslake (1986) suggests that many secondary school pupils in England fail to move beyond the idea that a fraction is a part of a whole. She suggests that this is related to Hart's (1980) report that there are secondary level students who avoid fractions and deny that it is possible to divide a number by one that is larger than it. Similar conclusions have been reached about pupils in Canada and the United States (e.g. Behr, Lesh, Post, & Silver, 1983; Chaffe-Stengel & Noddings,1982; Larson, 1980).

It is likely that one cannot accept the idea that there are numbers rendered by the operation of dividing one numerlog by another without the ability to use numerlogs to represent cardinal values, an ability in its own right that takes time to develop (Fuson, 1988; Gelman & Greeno, in press). Gelman & Greeno (in press) outline a learning sequence wherein the count words first re-represent the numerons that represent the numerosities of counted sets. Informally, this amounts to saying that the numerlog "five" represents the numeron obtained in counting sets of five. "Six" represents the consequence of counting a set that has one more item than do sets of 5, and so on. Although knowledge of the numerlogs for the positive integers marks an advance, there is no denying that it is far from what one means when one talks about using "five" or 5 as mathematical entities unto themselves, as we do when we say that 5 has the inverse (-5). In fact, such knowledge is at best a prerequisite for learning what Saxe (1988) refers to as a culture's orthography for number.

Learning about fractions involves more than working with numerlogs as cardinal numbers. More generally, learning about fractions seems to depend on being able to use, in mathematically meaningful ways, conventional mathematical terms, tools and symbols, including: the number line; non-integer numerlogs like "one half"; numerals -- the written symbols, or what we will call numergraphs, for corresponding numerlogs; and the notational systems for writing fractions (non-integer numergraphs) as the result of division or in terms of the decimal system.

Thus, the mastery of fractions requires the learning of principles that go beyond those implicit foundational ones available to the young child. It also seems to require the use of a language mediated system (See also Hiebert & Wearne, 1986).

In the absence of implicit principles for dealing with fractions, young learners might "overgeneralize" their counting principles and produce a distorted assimilation of the instructional data on fractions to an implicit theory of number that cannot handle such data. For example, if they do assimilate their understanding of the language of fractions to the implicit system of mathematical principles available to them, they should "read" fractions, or non-integer numergraphs, as if these are representations for the counting numbers as opposed to ways consistent with the principles underlying fractions. Thus, for example, they might choose $\frac{1}{4}$ as more than $\frac{1}{3}$, or when asked to place $1\frac{1}{2}$ circles on a number line, they might decide they have "two" things and place the stimuli at the position for 2 on the line.

The study of children's knowledge of fractions serves two goals: It serves as an entry point for assessing limits on the young child's knowledge of mathematical principles, and it provides a way of considering how young children use the symbolic tools of mathematics. These two goals converge on a theoretically and pedagogically important question. Since the formal principles that define the relations between multiplication and division include ones that are not included in those for addition -- the distributive law is not a law of addition but a higher-order one -- in what sense, if any, can mathematically meaningful learning about fractions build on what is already intuitively given?

The Main Study

Background

Studies of how school-aged children work with rational numbers have focused on children who are already half way through elementary school or even in high school (e.g. Behr, Wachsmuth, Post & Lesh, 1984; Hiebert & Wearne, 1986; Kerslake, 1986; Nesher & Peled 1986).

We know of none that have targeted children in their first few years of school, presumably because early math instruction does not focus on this topic. Since we were as much interested in how children would interpret reasonably novel as well as familiar data in this domain, our first study focused on Kindergarten (N=16, mean age 5yrs - 9mos), First Grade (N=12, mean age, 6 yrs - 9mos), and Second Grade (N=12, mean age 7 yrs - 9mos) children.

We thought it was especially important to focus on young children. For two conclusions emerge from relevant research with older children. First, even older children have a tendency to extend inappropriately generalizations they learn in school about the integers to other numbers. For example, they seem to believe that they can assume that the product that results when one multiplies two numbers is always larger than either of the original numbers -- even when the numbers are fractions. (e.g. Greer, 1987; Hart,1981). Note that this principle is valid for the *addition* of positive rationals but not for their multiplication. More generally, many secondary school students lack a mathematical understanding of division and multiplication (e.g., Fishbein, Deri, Nello, & Marino,1985; Vergnaud, 1983), a fact that cannot help but make one wonder how they could possibly understand fractions. These findings led us to think that it might be exceedingly hard for teachers to penetrate the child's spontaneous theory of number, despite lessons designed to do so. Before considering such a possibility one has to determine whether the proposed initial theory influences how children deal with their first encounters with fractions. If the young do bring with them the idea that numbers are what one gets when one counts things, they might start building, at an early age, erroneous representations of data meant to exemplify alternative notions of what numbers are about. These representations could, in turn, stand in the way of children's correct interpretation of later lesson plans on fractions.

Children in the early grades receive some instruction about fractions. Since our sample was interviewed at the end of their school year, even the Kindergarten children had had some experience with a standard number line and representations of $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{3}$. Experience with these non-integer numergraphs included at least their spoken forms, written examples of the corresponding phrases, their numerical or numerigraphic representations, and appropriately labelled and marked measuring cups. The children in the first and second grades had more experiences like these, both in terms of classroom presentations and testing opportunities. Therefore, although their curricula did not delve into the conceptual and arithmetic characteristics of fractions as numbers, the children were offered some relevant data about the nature of fractions. Our study was designed to provide information on how these children interpret such offerings.

Design and Procedures

Plan of the study : Children who participated had signed permission from their parents. They were interviewed in a quiet room away from their classrooms. For all but a few children, the interviews were conducted on three different days. The first and second days of the interview were separated by at least two days (but not more than a week). The third day of the interview could occur anywhere between one and two months after the second. Exceptions to this timetable were forced by the approaching end of the school year. Several children were presented with their third day items within one to two hours after they finished their second day items because they were about to go on their summer holidays.

The design of the study included a pretest, a five-phase placement interview, and a follow-up battery. The items for the first two placement phases were presented without hints. Each successive phase after these introduced more and more relevant mathematical information and offered more detailed mathematical descriptions about the props in the task. This meant that we provided successively more hints about the nature of the task as we moved to the use of more and more explicit mathematical language.

The decision to use an interview that included ever more explicit hints was based on two considerations: First, we did not want to conclude that children lacked competence without trying to maximize

understanding of the instructions and the setting. We were especially concerned that our youngest subjects might need some guidance in the interpretation of key words used in the task, which is why we provided more and more clues regarding the meaning of key terms. (Gelman & Greeno, in press).

Second, we anticipated that the way children would respond to our hints would be theoretically informative. In other studies we have found that hints are differentially effective, depending on how much mathematical competence a child brings to the task to start. For example, in our study of how Down's Syndrome and normal preschool children solved a novel counting problem, the majority of the Down's Syndrome children failed to benefit from hints, even when these were very explicit (Gelman & Cohen, 1988). In contrast, the preschool children improved their solutions, even when simply given the chance to try again. Other findings in the study converged on the conclusion that the preschool children were better able to take advantage of hints because they had a more principled understanding. Some, albeit incomplete, knowledge enabled them to take advantage of subtle hints, to recognize their own errors and to try again. Of interest here is whether a similar pattern will emerge: will some children benefit from hints and others not? If so, we can ask whether this is due to a related difference in fraction-relevant knowledge and/or the ability to apply it. For example, it might be that initial levels of success on the fraction placement task will be related to the ability to interpret the terms and symbols used in later phases.

All sessions were taped for later transcription.

Details about the study : Table 1 presents the organization of the study which included a pretest, a 5-phase fraction placement interview, and a follow-up set of items. The pretest questions and interactions familiarized children with "our special number line". When children came into the room, they saw 'The Count', a puppet from the television program Sesame Street, sitting in the middle of the table alongside the folded-up number line. They were told that The Count had come to visit them at school because he wanted to learn new things about numbers.

TABLE 1: SEQUENCE OF ITEMS IN THE FRACTION PLACEMENT STUDY

PART OF STUDY	INSTRUCTIONS AND/OR TASKS
DAY 1 OF INTERVIEW	
PRETESTS	
A. SOME RELEVANT VOCABULARY	-E Shows S 1-1/2 (1/2,1-1/4, 1/4,1/3) circles and asks "How many"? -If necessary, directed hinting.
B. INTRODUCTION OF NUMBER LINE	-Talk about the "special number line", circles instead of numbers. - S asked to show where 1&3 are. -Then asked where 2 goes; what would be before 1 and after 3. -Directed hinting, if necessary. -Talk about order/next, more/less, same. -S places N (1,2,3,&4) circles.
BEGIN FRACTION PLACEMENT PHASES	
PHASE 1: NO DEMONSTRATIONS	-S is asked to place and explain for 1-1/2, 1-1/4, 1-1/3. -E repeats 1-1/2 if S wrong on it; give S feedback if necessary.
PHASE 2: NO DEMONSTRATIONS	- Same as Phase 1, except values 1/2, 1/3, 1/4. - Repeat 1/2 as above if necessary. E asks if fractional parts are "same" or "different".
PHASE 3: BEGIN WITH HINTS END WITH SOME HINTS, ARITHMETIC & COUNTING	e.g. hint: *"We can put this (1/2) on a number line like this between 0 and 1 because it is less than 1 and more than 0."*, ...etc. - S now asked to place 2-1/2, 1-1/4, 1/3. (E does not identify values) - Ending items: E asks about numbers between 1 & 2 and 0 &1; S counts and solves X+0 (where X is an integer), 1/2+1/2, 1/2+0.
END DAY 1	

TABLE 1 CONTINUED:

PART OF STUDY	INSTRUCTIONS AND/OR TASKS
DAY 2 OF INTERVIEW (2 - 7 DAYS AFTER DAY 1)	
PLACEMENT PHASES CONTINUED	
PHASE 4: BEGIN WITH REVIEW. E LABELS ITEMS WHEN THEY ARE PRESENTED. END WITH ORDERING QUERIES ABOUT LABELLED VALUES.	-S places 1,2,3 for review. -" This is 1-1/4 (1-1/3). Where does it go? -"Which is more, 1-1/4....?"
PHASE 5: ALL ITEMS LABELLED & COMPARED IN SEQUENCED PAIRS. S ASKED WHICH IS MORE OF TWO WRITTEN PAIRS (NOT LABELLED)	-S places and explains and answers re 1/2 & 1/3. -More questions with 1/2 vs 1/4; 1/56 vs 1/75.
END DAY 2 OF INTERVIEW	
DAY 3 OF INTERVIEW (BETWEEN ONE AND TWO MONTHS LATER)	
FOLLOW-UP	-S reads 1/2,1/4; Which is 1/2?; Which is more? -Talk about measuring cups; More ordering questions about 1/2 & 1/4.
END OF INTERVIEW	

As can been seen in Table 1, to start the pretest, a child was first shown, one at a time, displays of $1\frac{1}{2}$, $\frac{1}{2}$, $1\frac{1}{4}$, $\frac{1}{4}$ and $\frac{1}{3}$ circles and asked "how many" were present. If a child could not name $1\frac{1}{2}$ or $1\frac{1}{4}$, the experimenter pointed to the whole circle and said "This is one circle". Then, while pointing to a part of a circle ($\frac{1}{2}$ or $\frac{1}{4}$), she asked "What's this?" The part was correctly identified for children who could not answer.

After this introduction to relevant terms, the experimenter unfolded her "special number line" schematized in Fig.1A. It can be seen that our number line did not use cardinal numergraphs to represent the cardinal values for successive integers. Instead, the line had one black circle where the integer 1 should have been, and three black, same-size circles at the place for 3. The number line display was 4'4" long and each circle was 4.75" in diameter. The line was intentionally long; it was meant to cover the length and then some of work tables found in schools.

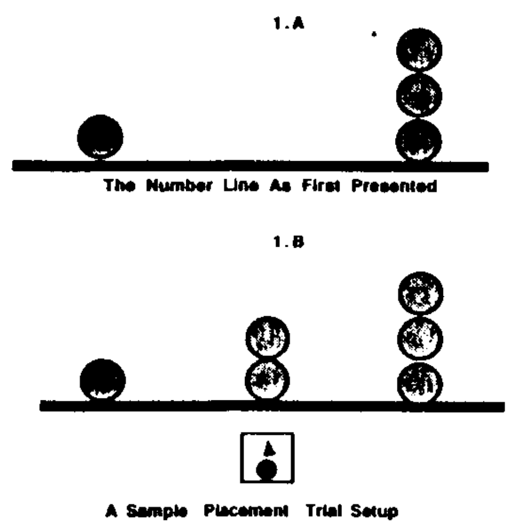

Figure 1: Schematic Representations Of The "Special Number Line" During Pretesting (1.A) And A Placement Trial (1.B).

Once unfolded the child was told that "our special number line does not have numbers written on it but shows where the numbers should be" and then asked to show us where "1" (and then,"3") were. This done, the child was handed two black circles and asked to show where to put "2". Then, while pointing to the left side of "1", the experimenter asked the child what number came before 1; similarly the child was asked what came after 3.

The pretest continued with talk about the ordering relations between the whole numbers, e.g. " Is 4 more than, less than, or the same

as 3; is 2 more than 1, less than 1 or the same?" If necessary, feedback was provided since we wanted to encourage children to think of ordering numerical values. Then, to end the pretest, children were told "The number line shows the numbers in order". To show us that they knew "how our number line works", they were then asked to place sets of N (1, 2, 3, or 4) small circles (ones much smaller than those on the line) "where they belong".

The pretest phase was followed by a five-phase sequence of fraction placements and related test-items. A sample fraction placement trial is shown in the bottom half of Fig.1. On such trials a child's task was to put each test display below the point on the line "where it belonged" and then explain why it went there.

Throughout the placement phases the color of the whole and fractional parts of circles varied (red, yellow, blue) across, but not within, a display. For example, all items in the $2\frac{1}{2}$ display were red, those in the $1\frac{1}{2}$ display, blue. Size of circle was held constant across and within displays.

No hints were provided during either of the first two placement phases. In phase 1 displays contained $1\frac{1}{2}$, $1\frac{1}{3}$, and $1\frac{1}{4}$ circles. The second phase displays were made of just these fractional parts, i.e., $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of a circle. Once Phase 2 was done, children were asked how the test items for that phase were the same or different.

As indicated, no hints were provided during either of these first phases. However, if children erred on $1\frac{1}{2}$ in Phase 1 or $\frac{1}{2}$ in Phase 2, these items were repeated at the end of that phase. As during the pretest, children now were asked whether the numerical value represented was equal to, more than, or less than a whole number value.

The experimenter started phase 3 by placing $\frac{1}{2}$ circle between the positions for 0 and 1, and saying "we can put one half between zero and one because it is more than zero and less than one". A similar trial followed with a display with $1\frac{1}{2}$ circles. Finally, the child was told "We can count $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2". Then the child was asked to place the test items ($2\frac{1}{2}$, $1\frac{1}{4}$, $\frac{1}{3}$) and explain their placement. The experimenter did *not* use the corresponding numerical descriptions for these test items.

The experimenter resumed talking about number at the end of the phase-3 part of the placement interview. Then the child was asked whether there were numbers between 1 and 2 (0 and 1), and if so how many there were. Denials led the experimenter to ask if $1\frac{1}{2}$ (or $\frac{1}{2}$) was such a number. The opportunity to count by rote and solve *verbally* presented arithmetic problems ended the interview for the first day.

Phase 4, which occurred at least 2 days, and as much as a week, after the previous placement phase, started with a brief reminder about the "special number line". For all subsequent test and demonstration items, the experimenter now used the terms that corresponded to the numerical values instantiated by the displays. Children were questioned extensively about the ordering relations of these as well as the ordered positions the corresponding stimuli should assume on the number line.

Phase 5 was much like phase 4 and came right after it. These phases differed mainly in terms of the display values presented -- one difference that parallels the difference between phases 1 and 2. As indicated in Table 1, for these last two placement phases, children had but two test displays per phase. For all of the previous ones they received 3 test displays per placement phase.

Notice that the end of phase 5 included some items designed to assess the extent to which our subjects interpreted the task as one that had something to do with counting things. The follow-up phase, a month to two later, yielded data on how the children read non-integer numergraphs. Finally, as can be seen in Table 1, the design included some arithmetic items so that we could assess whether there is a relationship between arithmetic skill and level of success on the fraction placement task.

Results

By the end of the pretest, all children could place sets of 1, 2, and 3 at the correct whole number positions. Their subsequent responses to the fraction placement target items were coded for each phase of the testing. Inspection of individual patterns of responding across phases 1 and 2 (i.e. before hints were offered) revealed four patterns of response categories:

(1) *Correct (At Least 50%)* : There were children who placed items so as to integrate the ordered positioning of fractional parts and whole circles and did so without feedback. Children who did this on at least 3 of their total of 6 phase 1 and 2 trials were scored as Correct. We did this even if children were not perfect because those who did not meet the at-least-50%-correct criterion responded differently than those who did. Therefore all other patterns of phase 1 and 2 responses were assigned to one of the following categories.

(2) *Parts Alone Rank Ordered* : Some children neglected any whole circles in a display and responded as if they simply rank ordered the relative sizes of the parts. For example, one child placed $1\frac{1}{4}$ at "1", $1\frac{1}{3}$ at "2" and $1\frac{1}{2}$ between "2" and "3". Such responses map relative amount of area to relative length without regard to the size of the interval between successive points on the number line.

(3) *Whole Number Placements* : Children in this category used counting strategi[illegible]e placed test displays as if they had counted the number of separable parts. For example, several children placed the display with $1\frac{1}{2}$ circles at "2"; and all displays with $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ circles at "1". Other children responded as if they ignored the fractions of a circle and simply counted the remaining whole circles. Such children also placed the preceding list of stimuli at either "1" or "0".

(4) *Others* : All response patterns that differed from the above three were coded in this category. For phases 1 and 2 these included

those where children placed successive test displays from left to right, put each test item at a different position without any concern for order, or generated sequences that we could not decode. Once the experimenter began to show children where to put displays containing one half of a circle (during phase 3), some started to mimic her. Mimics simply placed nearly all displays that had *any* parts on them halfway between two whole number positions. That is, they even placed displays containing $\frac{1}{3}$, and $\frac{1}{4}$ at half-way points between successive instantiations of whole numbers.

Number Line Placement Results

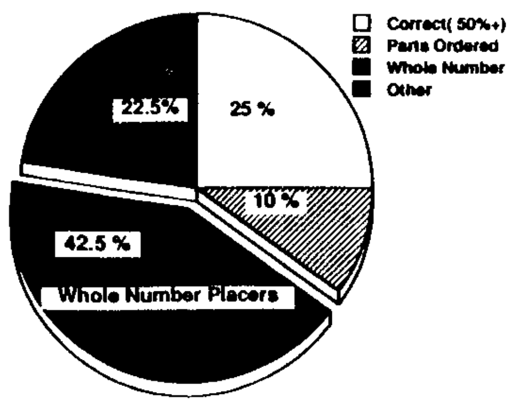

Figure 2. Pattern of Placements During Phase 1 and 2

Since the first two placement sessions were run without feedback or hints, these sessions provide baseline data on how children interpreted the task. As shown in Fig. 2, only 25% of the children were able to use the number line correctly on at least half of their trials. The

predominant tendency of those in the study was to place the test displays as if they had counted the items therein. More than 40% percent of the children did this. Another 22.5% produced solutions that were not discernibly task-relevant. Finally, a few children mapped their ordering of the relative size of fractional parts to ordered points on the number line. If they did this, they ignored both distance and the position of the whole numbers as instantiated by the number of circles at a point.

The mean ages of children in the Correct, Only Parts, Whole Number, and Other groups were: 7 years-6 months, 6 years-0 months, 6 years-3 months, and 7 years-3 months, respectively. More Kindergarten children were placed in the Whole Number and Only Parts Group; children in Grade 1 tended to fall into the Other group; those in Grade 2 were more likely to be in the Correct group. ANOVA's revealed reliable effects of grade ($F_{3,36}=6.12$, $p=.002$) and age ($F_{3,36}=6.62$, $p=.001$). Post-hoc Scheffe tests indicated that children in the Correct group were developmentally more mature than those in either the Order Parts ($F=3.35$ and 3.42, for age and grade, respectively, $p<.05$) and Whole Number (Counters) groups ($F=4.89$ and 4.74, for age and grade, $p<.05$). Differences between other groups were not reliable, including those between the Correct and Other groups.

Although there was a correlation between between developmental level and the way children responded to the placement displays, it was not large. The *r*'s between group assignment and age or grade were .26, and .22, respectively. In a factor analysis of the data, both variables loaded exclusively on one of two factors. Group assignment carried the weight of the second factor. What follows considers what else besides a child's developmental status might have contributed to success level on the placement task. In some of these analyses the four fraction placement groups described above are collapsed into two: *Correct Placers* and *Incorrect Placers*. Subjects in the original correct group make up the former, those in the remaining three groups make up the latter.

PP1 (31).max

Related Abilities

Initial Responses to the Novel Number Line.

Table 2 summarizes how children assigned to the different fraction placement groups dealt with the special number line. It can be seen that all groups did rather well on these pretest items. Children in the Correct placement group were perfect on all their pretest items and although children in each of the remaining groups had some problems with these items, the differences are not that large. Given our hypothesis that children would assign whole numbers to a correct corresponding relative position on the number line, we would have been surprised had these differences been greater.

Table 2: How Children in Different Placement Groups Responded to Pretest Items

Group and Responses Patterns (Based on Phase 1 & 2 Placements)		Test Items: Knows Zero (or Nothing)	Test Items: Places (Knows) 2,3,4	Test Items: Ordering (Mean, max= 4)
		SCORE		
I	Correct Placements on At least 50% of trials (n=10)	100 %	100 %	4.00
II	Rank Ordering of Size of Parts of Circle along Line (n=4)	50 %	100 %	3.75
III	CountsThings and Uses Whole Number Positions (n=17)	75 %	89 %	3.12
IV	Other (n=9)	78 %	89 %	3.78

Differential Effects of Hinting.

To determine how the different groups of children responded to hints, we looked at who improved as a function of hints. A child was scored as improved if they were in the top group (Correct on at least 50% of their trials) to start and their percentage of correct trials increased. Children who started in another group and then met the criterion for the Correct group, were also scored as improvers.

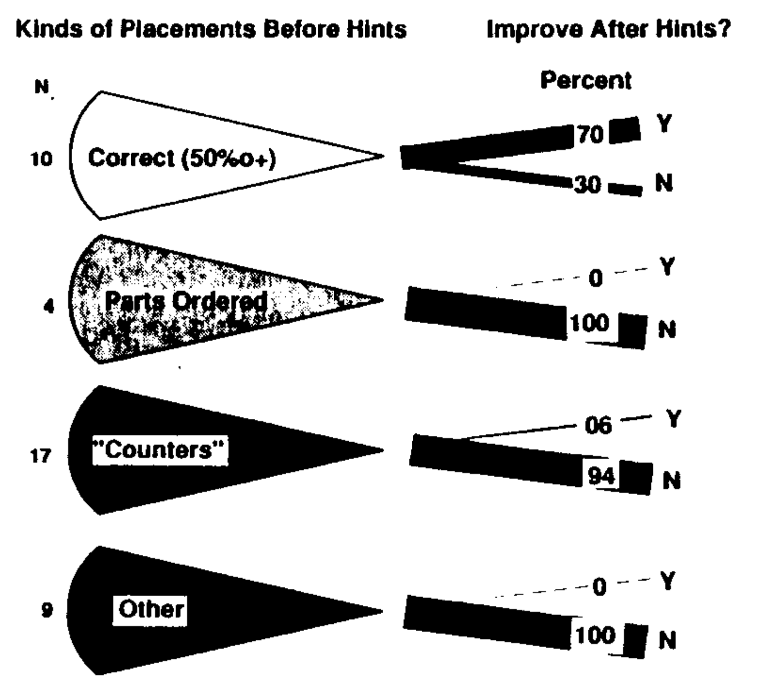

Figure 3: Percent Ss in Pre-Hint Placement Groups Who Improve After Hints

Figure 3 summarizes the effects of hints. It shows that children who did well to start tended to improve when given hints. In contrast, those who thought the task was about counting numbers or the ordering of size of parts resisted our hints. So did children who used other task-irrelevant strategies and therefore started in the Other group. Hence, children who started as non-fraction placers continued to respond as non-fraction placers, continuing to apply their initial solutions or moved to ones that were even less consistent with the task requirements.

If anything, hinting had a deleterious effect on children who were not in the Correct group to start. The general tendency for the Counters was to stop counting and to mimic the experimenter once hints were introduced. A similar tendency to place all fraction items at some common point between two whole number positions characterized the children who started out in the "Other" category. In contrast, no child who was classified as Correct before the hints were introduced simply imitated the experimenter's demonstrations although one child in this group did switch to a counting solution after the hints were introduced. All others in this group either maintained their initial levels of performance or improved.

In our introduction we considered two possible effects of hints. These simply might serve to clarify the intent of the task. If so, all children who erred to start should have done better after the hinting phases. Alternatively, hints might interact with children's initial level of understanding, helping only those who already had some understanding of the concepts involved in the task. The second of these possibilities best describes the results.

Ability to Read $\frac{1}{2}$ and $\frac{1}{4}$.

Next we consider whether both the differences in initial success levels and tendency to benefit from hints went along with broader differences in the ability to deal with both fractions and mathematical symbols.

TABLE 3: HOW EACH CHILD IN EACH PLACEMENT GROUP READ 1/2 AND 1/4

FRACTION SUBJECT AND GROUP	1/2	1/4
CORRECT		
1A	1 plus 2, 3, 5	1 plus 4
2A	half, one half	one fourth
3A	one half	one fourth
4A	a half	one four
5A	1 and a half	1 and a fourth
6A	a half	one fourth
7A	1 and a half	1 and a fourth
8A	1 and a half	1 and four quarters
9A	one quarter	one fourth
10A	1, 2	1, 4
ORDER PARTS		
1B	1 and 2	1 and 4
2B	1 line 2	1 line 4
3B	1, 2	1,4
4B	1 plus 2	1 plus 4
WHOLE NUMBER (COUNTERS)		
1C	1 and 2	1 and 4
2C	1 and 2	1 and 4
3C	1, 2	1, 4
4C	twelve	fourteen
5C	1 plus 2	1 plus 4
6C	2,1 "saying backwards"	4,1 "saying backwards"
7C	1 and a 2	1 and a 4
8C	1, 2	1, 4
9C	four	five
10C	1 and 2	4 and 1
11C	1 and a 2	4, 1
12C	1 and 2	1 and 4
13C	1, 2	1, 4
14C	1, 2	1, 4
15C	3	5
16C	1 and 2	1 and 4
17C	1, 2	1, 4
OTHER		
1D	1 and 2	1 and 4
2D	1 and 2	1 and 4
3D	1 and a half	1 and a fourth
4D	1 and 2	1 and 4
5D	one half	one fourth
6D	1 and 2	1 and 4
7D	1, 2	1, 4
8D	1, 2	4, 1
9D	one half	one fourth

As is evident in Table 3, there was a strong relation between pre-hinting placement success and the ability to read the non-integer numergraphs $\frac{1}{2}$ and $\frac{1}{4}$. Almost none of the children in any of the "Incorrect" groups could read the test items appropriately. Instead they had a potent inclination to **misread the fractions as two integers**, e.g. "1,2"; "1 and 2"; etc. In addition, many of these children read the division symbol as either "and", "plus", or "line". Finally, some children behaved as if they had been given an addition problem; when asked to read $\frac{1}{2}$ and $\frac{1}{4}$, they answered "three" and "five", respectively.

To say children failed to read correctly the non-integer numergraphs is to simplify the matter. The results suggest that the children once again revealed an assumption that the task had something to do with counting things and adding whole numbers. Where we assumed we provided inputs about fractions, they behaved as if they were offered further cases to which they could apply their view that numbers are what one gets when one counts and adds the results of counting. What follows is consistent with our interpretation.

More on How Fractions are Treated as Count Number Inputs.

Do Young Children Think $\frac{1}{4}$ is More than $\frac{1}{2}$?

If our test items detected the assimilatory power of the idea that numbers are what one gets when one counts, our subjects might be expected to choose $\frac{1}{4}$ as more than $\frac{1}{2}$. Since we saw that they did not treat these expressions as fractions, they might fall back on comparing only the 4 and 2 and therefore say that the symbol $\frac{1}{4}$ stands for a greater value than does the symbol $\frac{1}{2}$.

Children in the Correct group do somewhat better than those in the Incorrect groups on the "which is more" task, but only because, as a group, they responded randomly. Four subjects (of 10) chose $\frac{1}{2}$ as more than $\frac{1}{4}$, and three chose $\frac{1}{56}$ as more than $\frac{1}{76}$. In contrast, the Non-Correct subjects had a reliable bias to select the reciprocals with the larger denominators as more ($\chi^2_1 = 11.27$, $p < .001$ and 8.06, $p < .01$, respectively).

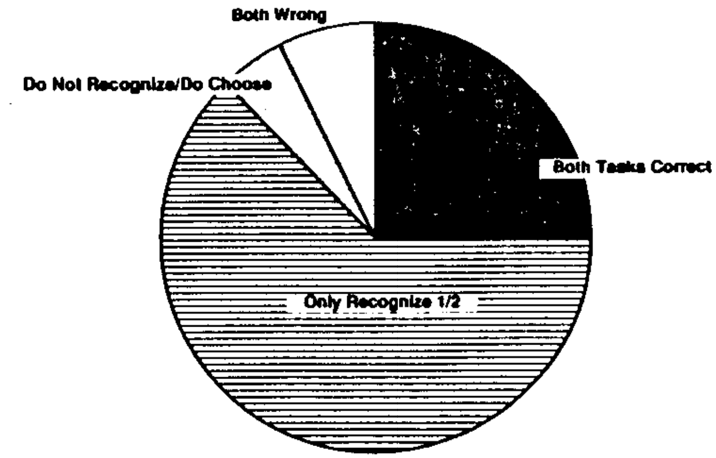

Figure 4. Ability to Recognize 1/2 and Choose it as More Than 1/4.

One possible explanation for the "$\frac{1}{4}$ is more than $\frac{1}{2}$" result is that children could not recognize $\frac{1}{2}$ when asked to point to the "one that shows $\frac{1}{2}$", this because of the way they read these expressions out loud. The relevant data are shown in Fig. 4 which shows that well over 75% of the sample could point correctly to $\frac{1}{2}$ when shown both $\frac{1}{2}$ and $\frac{1}{4}$, i.e., most of the children could pass an item like ones on multiple choice tests. The problem is that only 25% of them could also employ these symbols or their referents in a manner consistent with their mathematical meaning.

Other Arithmetic Tasks.

We included two other problems where we thought children might make a counting-based error, $(\frac{1}{2} + \frac{1}{2})$ and $(0 + \frac{1}{2})$. Fig. 5 shows the extent to which children did so. The bias to do this on the $\frac{1}{2}$ vs $\frac{1}{4}$ item is also plotted. Table 4 summarizes error tendencies on these tasks as a function of fraction placement skill.

TABLE 4. PLACEMENT ABILITIES AND ERROR TENDENCIES WHEN CHILDREN CHOOSE THE GREATER OF $\frac{1}{2}$ and $\frac{1}{4}$, ADD $\frac{1}{2}+\frac{1}{2}$, and ADD $\frac{1}{2}+0$*

	Placement Group			
Error Kind/ Tendency	Correct (n=10)	Parts (n=4)	Whole Number (n=17)	Other (n=9)
Percent Errors For All Trials	30	75	78	67
Percent of All Trials with Count-like Errors	30	58	55	33
Percent Ss Who Contribute At Least One Count-Like Error	60	100	94	78

*A count-like error was coded as when S said that $\frac{1}{4}>\frac{1}{2}$, $\frac{1}{2}+\frac{1}{2}$ was "2" circles or "2 and a half"; and $\frac{1}{2}+0$ was "one" or "one and a half". These responses contributed to the figures in the bottom two rows. Those in the top row are based on all errors.

Note that answers of the form $(0+\frac{1}{2}) = \{1, 1\frac{1}{2}\}$ are relatively infrequent, especially as compared to ones of the form that $\frac{1}{4} > \frac{1}{2}$ or $(\frac{1}{2}+\frac{1}{2}) = 2$. Such a pattern of results is consistent with both Evans' (1983) and Wellman and Miller's (1986) conclusion that children quickly master a purely procedural algorithm for dealing with zero addition problems, i.e. repeat the non-zero value whenever it is to be added to zero. To do this one need not know that zero is the identity element under addition and it seems that children as young as ours do not (Wellman & Miller, 1986).

In Table 4 we can see that Correct Placers were less likely to make errors on the arithmetic problems summarized in Fig. 5. Still, if we consider the extent to which children made a count-like error on at least one of the three problems, we see that even in our most skilled placement group, 60% of the children did this. It would seem premature then to say that they had a deep understanding of fractions.

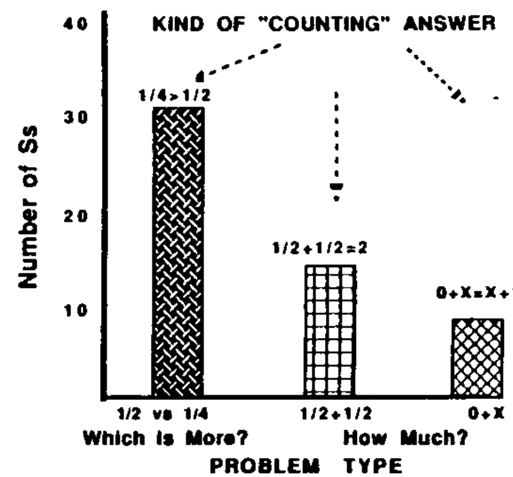

Figure 5. Number Of Children Who Make Count-Like Errors On Different Problem Types

How Much Do Correct Placers Know About Fractions?

How much credit should we give the children who mapped our test items onto the number line and did well on a variety of other tasks? In particular, can we say they understand the isomorphism between points on the number line and numbers, that fractions effectively make the system of numbers continuous by filling in the gaps between two integer points on a line [1]? To grant understanding of this fundamental mathematical principle, we at least need evidence that children know there is a very large (actually infinite), number of non-integer numergraphs between any two integer points on the number line.

[1] By effectively continuous, we mean two things: 1) Between any two rational numbers, there is a denumerable infinity of rational numbers; and 2) For any point on the line there is a point corresponding to a rational number that lies arbitrarily close (the principle underlying Dedekind cuts).

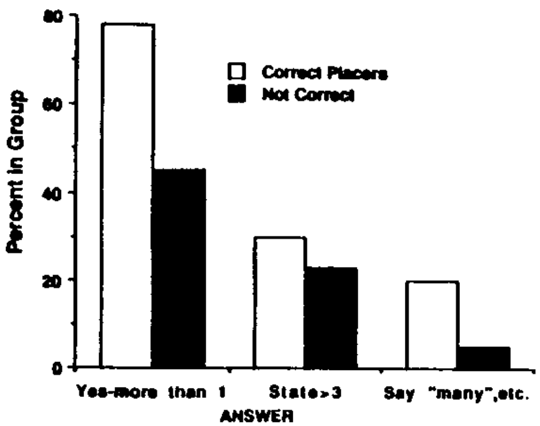

Figure 6. Do Children Think There are Numbers Between the Integers and If so, How Many Do They Say There Are?

Figure 6 shows how children answered questions about whether there are numbers between 0 and 1 and how many. Those who were scored as Correct on our placement task were more inclined to agree that there are numbers between 0 and 1. Still, the bias of the children was to say that there are about 3 fractions between the integers. This suggests they were simply recalling the three fractions we used during testing. Very few children in any group could tell us that there were "many", "hundreds", or "thousands", or "as many as you want", possible numbers between two integers.

In brief, even our best subjects were inclined to interpret arithmetic problems with non-integer values as ones with count numbers. They were inclined to answer that there were but a few numbers between two integers. Of course, we expected our subjects would make such errors, because they were all too young to have received much instruction about fractions. However, it is not clear that instruction overcomes these initial beliefs. Findings from a variety of studies highlight the possibility that still older children lack a mathematical understanding of why a fraction is a

number. For example, Kerslake (1986) notes that very few pupils in her secondary school sample were either familiar with or accepting of the division aspect of a fraction. Instead they thought of fractions in terms of the number of parts in a given whole, a conclusion that is also reached by Silver (1983) on the basis of his interviews with college students in a teacher preparation course.

We agree that the tendency to think about fractions in terms of particular examples of part-whole representations is a barrier students have to get beyond before they can understand fractions. However, it seems that the student in Silver's study who answered that $(\frac{1}{2}+\frac{1}{3}) = \frac{2}{5}$ has yet another problem. Such solutions are reminiscent of children in the above study who said that $\frac{1}{4} > \frac{1}{2}$, because 4 is more than 2, and read such representations as lists of whole numbers. If Silver's subjects were inclined to do the same, that is view such marks on paper as representations of integers, then they would have to answer $\frac{2}{5}$, this because $1+1=2$ and $2+3=5$. In other words, the belief that numbers (and their numerlog representations) are what one gets when one counts appears to continue to influence performance on arithmetic problems well after students have had instruction about the nature of fractions.

A similar conclusion is reached about students' conceptions of decimal numbers. To quote Hiebert and Wearne (1986): "To our knowledge, all researchers who report data on ordering decimals ... suggest that many students have trouble judging the relative magnitude of decimal fractions if they have different numbers to the right of the decimal point (e.g., 1.3 and 1.295). Most errors can be accounted for by assuming that students ignore the decimal points and treat the numbers as whole numbers" (p.205). This generalization is supported by Hoz and Gorodetsky's (1983) data on how decimal fractions are read out loud, e.g. "seven" or "zero seven" for 0.7, "thirty-eight meters" for 0.38 m, "3 meters and 8 centimeters" for 3.8 m, etc. Hiebert and Wearne are careful to point out that this is not the only misconception that pupils have about fractions and the interpretation of the convention for representing them as non-integer numergraphs. Others develop as a function of input, for example, that the more numbers there are to the right of the decimal, the smaller the value represented (Sackur-Grisvard & Leonard, 1985; Resnick & Nesher, 1983). But the point still remains: these are hardly correct mathematical understandings of fractions. More importantly, it appears that even decimal fractions are assimilated to a theory of numbers based on the idea that "numbers are what you get when you count".

We present a few results from a second study of ours. These help make contact between those in the literature and our study with young children and introduce some questions for discussion.

The Questionnaire Study

Some Details about the Study

We were able to coordinate our interests in the way children think about fractions with the goals of the mathematics superintendent of a school district in a relatively large MidWestern suburban community, henceforth assigned the alias The Prairie School District. We were able to present, in written form, a number of items that were selected to (a) relate our findings to those in the literature and (b) assess older students conceptual understanding of fractions. Since the school district offered to include their gifted classrooms, we sampled from two ability levels, regular and gifted. The school system assigned children to these different groups, as early as the second grade, on the basis of overall aptitude and performance levels. Children in the gifted groups were in self-contained classrooms in the regular schools and moved as a group through the grades.

Figure 7 summarizes a sample of our data on what children in different grades and programs did with a battery of "which is more items". It is clear that their responses are consistent with those in the literature on fractions and decimals (See above for a review). The one result that stands out and that is perhaps new has to do with the gifted - regular comparison. Put simply, there is a large effect of this variable, so much so that gifted 4th grade classes outperform regular 7th and 8th grade classes. A similar result is revealed in the final table we present, which

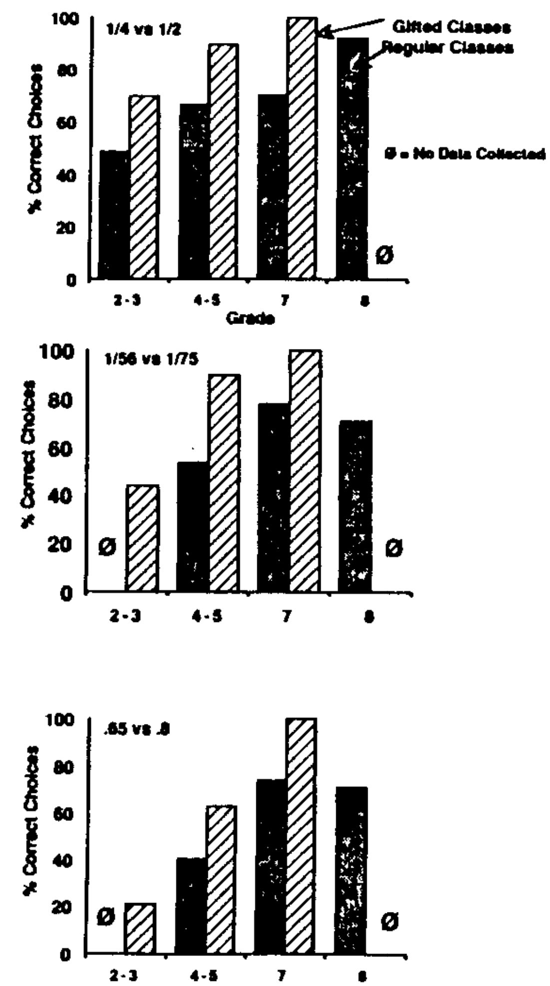

Figure 7."More" Choices as a Function of Grade and Class Kind

reproduces some of the answers children gave to our question "Why are there two numbers in a fraction?". We think the answers speak for themselves and make it clear that it is only the gifted children who can state the mathematical principles involved with any regularity.

TABLE 5: SAMPLES OF WRITTEN ANSWERS TO "WHY ARE THERE TWO NUMBERS IN A FRACTION?"

GRADE and SUBJECT NUMBER

Regular 4th and 5th Grade

1. "Because if there weren't 2 numbers then you couldn't have a fraction."
2. "It is two numbers"
3. "can't explain"
4. "To be eqivelent."
5. "Because a fraction is a part of something. Not the hole thing."
6. "Because you have to hav a denomonator and a numarator"
7. "1 for one certain color on the thing 4 for all the things in the group"

Gifted 4th and 5th Grade Class

6. "4 is how many piecies and 1 is how many you got"
8. "the bottom one is how many in the whole and the top one is how many are left a in the whole."
9. "The denmunator is the hole, the numirator is how many peices you have of the hole"
10. "...one of the numbers stand for how many thimes something is divided up into and the other how many are taken."

Regular Grade 7

11. "Because they wanted it that way."
12. " 1/4 1 explains how many you have. 4 explains how many there are"
13. "1 shows you how many you have and the other shows you how many are there"
14. "Because 1 number is how many of something you have and how many there are in a whole."
16. "Because it Broke in to fractions I GUESS"
17. "Because"
18. "because it is a portion of a number"
20. "you need a numberator & denominator"
21. "the top one explain how many of the Bottom one there are"
24. "It is less than one whole"
25. "to show both halfs"
26. "Because it shows how many items you have out of a given amount of numbers."

Discussion

The tendency to assimilate fractions to a conceptual scheme appropriate only for the positive integers appears as soon as fractions are introduced and persists at least into the secondary school level. Indeed, judging by Lave's (1988) work, it persists into adulthood in a large part of the population. This conceptual difficulty is manifest early on in the inappropriate placement of fractional parts on the number line and in a persistent misreading of the notation for fractions. It continues to be evident for many years in the misordering of numergraphs for fractions and decimals and an inability to talk coherently about the underlying principles. The evidence is clear that preschool conceptual schemes determine what many children extract from the curriculum.

It is significant that the above difficulties are much less evident in a selective sample of children -- even as early as the fourth grade. Many of these children order fractional numergraphs appropriately; they correctly assimilate what they, like all elementary school children, are taught. Moreover, they give appropriate answers to questions about underlying principles, even when these questions touch on matters that are not an explicit part of the curriculum. Many can write appropriate, well-formed answers to the question of why there are two numbers in a fraction.

These findings suggest that there are severe limits on what can be expected from purely situational instruction in mathematics. Such instruction may never induce many pupils to exceed the limits of the conceptual scheme they bring to the school environment. Mastery of fractions is a recognized watershed in the elementary curriculum because it requires students to transcend this intuitive conceptual scheme. They have to learn to apply the relational and combinatorial operations of arithmetic symbols that are not defined by either counting procedures or by language that refers to the intuitively given aspects of arithmetic reasoning. In Skemp's (1971) terms, they have to accomodate their ideas of what numbers are if they are to understand why fractions are numbers.

The correlation between early mastery of the ordering of fractions and the ability to talk coherently about the principles underlying the construction of fractions suggests that learning to apply language to the principles governing arithmetic operations may be crucial. Learning mathematics appears to go hand in hand with learning the language of mathematics. The language of mathematics does not share the same syntax as does English, or for that matter, any natural language. It has its own notational system and terms like *and, plus, or, add,* etc., have different meanings in the language of mathematics than do their cognates in a natural language. Could it be that we unwittingly put students at risk with respect their chances of learning mathematics if we do not focus on these differences? An answer to this question could contribute to our efforts to teach mathematical literacy. It might also provide clues as to how to teach children why fractions are numbers.

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