The Epigenesis of Mathematical Thinking

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The proclivity of young children to engage relevant environments actively helps explain how 3-year-old children in cultures that offer a variety of mathematical examples develop coherent understandings about natural numbers. A similar line of reasoning accounts for the development of other kinds of early cognitive accomplishments, such as understanding mechanics, people's minds, and the nature of animal action. This paper focuses on research findings in the number domain. The aim is to provide evidence that has shaped the view of developmental cognitive scientists about what young children are capable of and what this means for children's later learning. These findings set the stage for considering the conceptual changes that students must undergo to achieve the kinds of mathematical understandings that are the targets of school mathematics programs.

During the last 20 years, we have witnessed a remarkable growth industry in the study of mathematical cognition—be it with animals, infants, toddlers, or individuals who live in cultures with or without schools, in cities, in towns, and even in the remote ranges of the northern Congo. Given the ever-increasing and converging lines of evidence from various research sites, it is hard to escape the conclusion that there is a universal, domain-specific ability to learn about and reason arithmetically with the natural numbers, the count numbers.

As we shall see, this is likely the source of a problem we encounter in teaching mathematics and science. But first, we focus on the fact that the cumulative strength of the evidence encourages investigators to adopt the proposal that there are invariant arithmetic structures of mind that contribute to the widespread development of an implicit but principled understanding of natural numbers. This understanding enables individuals to count and to generate cardinal values that can be used in a structure that is something like addition and subtraction.

We come to the world of learning with a number-relevant mental structure that is comprised of skeletal principles for counting, for generating cardinal values, and for adding and subtracting the resulting cardinalities. Minds have an ubiquitous tendency to apply existing structures. No matter how skeletal the natural number

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skeleton may be, its presence leads us to seek out, attend to, and assimilate number-relevant data. Skeletal structures enable us to find, traverse, and interact with relevant learning paths that happen to be in our personal life spaces—including things in our physical, social, cultural, or mental environment that can help us develop the cognitive processes that lead to domain-specific knowledge about arithmetic.

From this perspective, we can think of young children as self-monitoring learning machines who are inclined to learn on the fly, even when they are not in school and regardless of whether they are with adults. The proclivity to engage relevant environments actively helps explain how 3-year-old children in cultures that offer a variety of mathematical examples develop a rather coherent understanding about natural numbers. A similar line of reasoning accounts for the development of other kinds of early cognitive accomplishments, such as understanding mechanics, people's minds, and the nature of animal action. Examples of research findings in the number domain are discussed herein with the aim to offer some of the evidence that has shaped the view of developmental cognitive scientists about what young children are capable of and what this means for children's later learning. These findings set the stage for an examination of the conceptual changes that students must undergo to achieve the kinds of mathematical understandings that are the targets of school mathematics programs.

It has been widely assumed that preschool children cannot possibly engage in anything like an abstract concept, that in fact, they do not have any abstract concepts and are bound by perception, prelogical, and even egocentric. Therefore, the proposition that children have principled understanding of the natural numbers may strike some as decidedly odd. To build the case for granting preschool children an organized understanding of at least some range of the natural numbers, samples of data are chosen from different kinds of research, data that have led a variety of people to concur that this is something we take an interest in and know about.

First, what do we mean by "counting?" In work with Gallistel (Gelman & Gallistel, 1978; Gallistel, 1992), we proposed that counting involves five principles, with the first three supporting the goal of achieving either nonverbal or verbal cardinal value of the collection being counted and the next two involving permission conditions for applying the first three. The five principles shown in Figure 1 are: (a) the one-one principle, (b) the stable ordering principle, and (c) the cardinal principle. Respectively, these principles mean that: (a) in a count of items, there must by one and only one tag per item, each of which is a unique tag; (b) the tags used must be such that they can be ordered stably over time; and (c) the last tag in the stable order list has the unique capacity of representing the cardinal value of the set, be this what Gelman and Gallistel dubbed a nonverbal *numeron* or a corresponding verbal tag. The fourth principle, the abstraction principle, captures the fact that (d) when one counts in a principled fashion, it matters not what it is you are counting so long as you think of the items as entities. Thus you can collect together for a given count all the great minds in the room, the pieces of furniture, the chairs, and the holes in the wall. Finally, the fifth principle, the order-irrelevance principle, says that (e) so long as you honor the first three "how-to-count" principles, it does not matter whether you skip around or in which order you count the items.

The proposal that young children understand counting and the related principles

Counting and Arithmetic Principles

- **COUNTING PRINCIPLES**
  - **HOW TO'S**
    - 1. One-one
    - 2. Stable order
    - 3. Cardinal
  - **PERMISSIONS**
    - 4. Order Irrelevance
    - 5. Item Kind Irrelevance

Numerons that stand for the cardinal value (N) of counted collection of discrete entities. If numerals mapped to counting list, cardinal count word re-represents cardinal quantity

- **ARITHMETIC REASONING PRINCIPLES**
  - **ADDITION & SUBTRACTION**
    - A/S applied to cardinal values
    - If A/S, relevant operation and N changes, cardinal value > or <
    - IF N-irrelevant operation, N is same (=) cardinal value

Figure 1. Counting and Arithmetic Reasoning Principles.

of addition and subtraction does not mean that they can state, let alone explain, what was just stated above. For sure, they cannot articulate the principles; indeed, it is a reasonable guess that neither could many adults. Instead, the claim is that children have an implicit knowledge of the kind that we all have of linguistic principles. If I say the word sequence, "Who did John see Mary and?" and then ask you if it is a sentence, you know it is not a sentence. Further, although you may not be able to say which constraints on sentence production have been violated, you can correct the sentence. You can make it into an acceptable English sentence. Such data indicate conceptual competence that is available to the mind, although it cannot be articulated (Gelman & Meck, 1986).

Findings like these led us to ask whether preschoolers could catch counting-principle errors in an error detection task. To do this we put a number of small items on a table. A researcher holding a hand puppet told a child the following story.

> This puppet, Mr. Horse, doesn't know how to count very well. Do you think you can help him learn how to count? He's just learning how to count. Sometimes he counts in ways that are O.K. and sometimes he counts in ways that are wrong. It's your job to tell him whether he was right to count the way he did or whether he made a mistake. Now remember, you have to wait until he is all done counting and until he has told us how many there are before you tell him whether he is right or wrong (Gelman, Meck, & Merkin, 1986, p.10).

The reason for these specific directions is that we don't want the child to say, "Wrong!" before the puppet has finished a trial. This was especially important on trials where the children watched as the puppet either skipped or double counted an item and then answered the cardinal question by giving the penultimate tag used. For example, there were six items on the table. The researcher had the puppet count, "One, two, three, four, five, six," then asks the puppet how many there are, and the puppet answers, "Five."

The critical question is whether young children can catch and correct the error. The answer is that they can. They say, "That's wrong, he should have said six." Not only can 4-year olds answer these items correctly, so can 3-year olds if we use set sizes of five or less. In this case, 85% of these young children were able to detect important errors in the principled application of counting. They also distinguish true errors from what we call *pseudo-errors.* For instance, when we start to count a set in the middle but come around and count all of the items, children can tell us that the latter is a silly way to count but not wrong.

Another kind of evidence comes from asking children to solve a novel counting problem that depends on their knowing about counting (Gelman et al., 1986). This is called the *order-irrelevance task* or *doesn't matter task.* Imagine that each of the five circles below represents a different small toy:

O O O O O

These circles could represent a small chair, a baby doll, a star, and so on. The child's task is to count in a special way. For example, we may point to the second item in the row and call it "one," then ask the child to count all the items. So the child, starting with the second item, might say, "One, two, three, four, five" while pointing to the second item, then the first, then the third, fourth, and fifth. Or, we may have the second item be "three," in which case the child has to start the count of the same display at a different item. The task continues until eventually the child is asked to make the second item "six", a number that represents more than the value of the set. This trial requires implicit understanding of the fact that *N* does not change simply because someone makes a counting error, and thus offers another way to get at whether children understand the relationship between counting, cardinality, and the interdependence of counting and arithmetic principles.

I will imitate a child answering during this task.

I (RG) say: "Ann, how many toys are on the table?"

Ann: "One, two, three, four, five."

RG: "How many?"

Ann: "Huh?" Then she points and says, "Five."

RG: "I have a puppet who likes to watch counting tricks, and this is the trick. He wants you to count them all but in a new way. Count them so that this item (pointing to the second item in the row) is 'one.' I bet you can do that."

Ann: "I can too." And then she counts, "One, two, three, four," starting with the second item in the row, proceeding through the third, fourth, and fifth items, and ignoring the first.

RG (pointing to the first item): "What about this one?"

Ann: "I left that one aside."

RG: "But I asked you to count all of them."

Ann: "But you said make this one!"

RG: "All right." At this point, with some indirect hints, we go on. "Make this two," I say, pointing to the second item.

And that's easy; that's a correspondence matter. Beginning with the second item, she just points and counts in order, "One, two, three, four, five."

In the next trial I say, "Can you make this a three?" while pointing to the second item.

And she says, "I think I can do that. I bet I can too." And she looks at them, then picks up the second and third items and switches their order. And counts, "One, two, three, four, five. Easy!"

This continues until we get to six, and then she does something really interesting. I say, "Can you make that a six?" while pointing to the second item.

And she says, "Hmmm . . . " She counts silently, pointing to each of the five items as she proceeds, then declares, "No!"

"Why not?"

"You need another one."

What is interesting in this response is that she is checking the universe of objects and is saying no, there aren't enough objects to solve the problem that's confronting me. In addition, she does a lot of self-correction, which we allow her to do. You cannot get this kind of performance, this forward movement, this epigenetic effort on the part of the mind, if there is not anything available to enable her to benefit from the constrained database with which she is presented. One way to make that point particularly clear is to contrast these 3- and 4-year-old children, who do very well on this constrained counting task, with mentally retarded children.

We had 10 children with Down's syndrome do the "doesn't matter" task (Gelman & Cohen, 1988). The ages of the children ranged from 10 to 13 years and their mental ages were at least equal to or more than groups of normal 4- and 5-year-old preschooler children. On the basis of a standard counting pretest, we were able to sort the Down's children into two groups, group A, consisting of two children whose counting ability exceeded that of the preschoolers in the study, and group B, consisting of eight children whose counting ability was comparable. Only the latter group are discussed in this presentation. For the record, you should know that performance of the two children in group A compared favorably with that of the preschool children.

As noted earlier, in some cases Ann would initiate a retrial. Figure 2B shows the tendency of children in the different groups to repeat the trial, either as a function of their doing it by themselves or our asking them to do it. The most important result here is that the group B Down's syndrome children seldom self-initiated a repeat, whereas there was a considerable tendency of the preschool groups to do so, especially in the 5-year-old group. These differences are related to the finding in Figure 2A that the preschool children used a greater number of strategies. Here then is another example of the considerable tendency of young children to self-initiate a repetition of a trial where, on their own, they take an error trial and redo it, not necessarily with the same solution. They won't necessarily say, "I know I've got it wrong," but they will just do it again, as if they know that their attempted solution did not come out right. See Gelman and Williams (1998) for further discussion of how available structures can support self-initiated practice and correction trials.

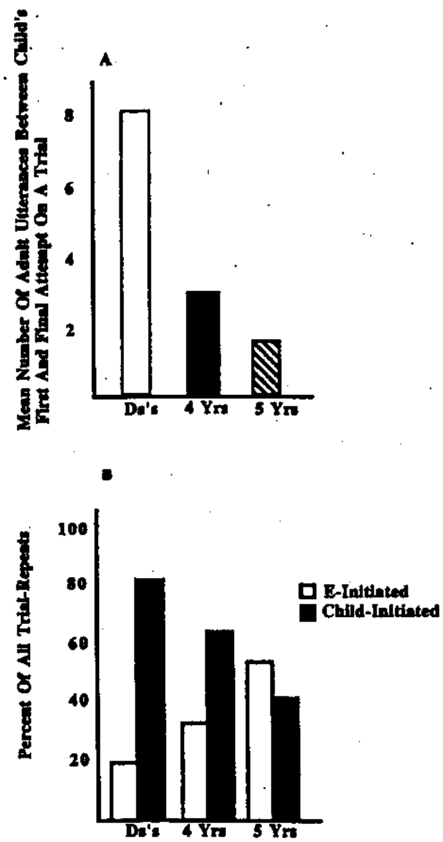

Figure 2. A: Mean Number of Different Strategies; B: Tendency of Children to Repeat a Trial Spontaneously or in Response to the Experimenter.

Figure 3 shows the number of probes experimenters provided on a trial, breaking these down into different categories of hints. As the figure shows, there is a large difference between the effectiveness of the hints for the Down's syndrome and preschool groups. Further, indirect hints sufficed for the latter groups. In contrast, hints involving the demonstration of a solution dominated our interactions with the children with Down's syndrome. However, showing a sample solution to the group B children did not help them improve.

The differential kinds and effectiveness levels of the hints add weight to the constructivist view that learning is not the result of simply practicing a particular input or the presentation of learning opportunities. Instead, learning depends on the existence of a mental structure that is at least incipiently available for interpreting the data, at which point practice becomes very useful. Such a structure allows for self-initiated practice trials, or self-interpretation, or mapping from the data to existing mental structures. Learners are active interpreters of inputs and use what-

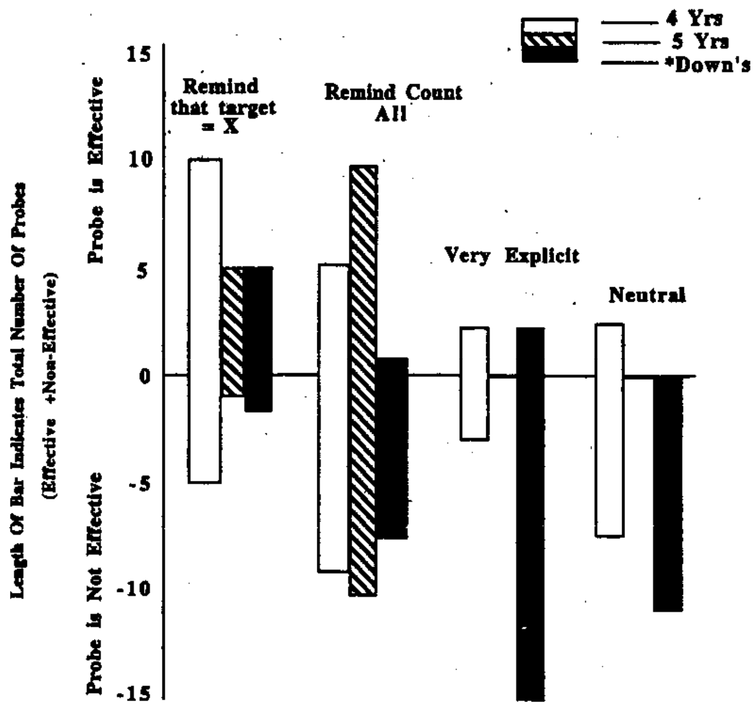

Figure 3. The Differential Effects of Hints.

ever they already know to make sense of novel inputs (Gelman, 1994; Glaser, 1988). All this is good. But there are theoretical problems created by such a model. For one, it is no longer possible to assume that a learner will interpret data as intended by a teacher, parent, more knowledgeable sibling, and so forth. To illustrate, consider the work conducted at the Please Touch Museum in Philadelphia (Gelman, Massey, & McManus, 1991). We followed parents and children around the museum to see what they did at an exhibit called "How Many?".

The Please Touch Museum is for preliterate children. The clients are children under 7 years of age who cannot read. Many of their parents do not read to them. The number of parents who interacted with their child at the display was not large. Of the 30% of parents who did, only 33% asked their children the "how many?" question. Within that 33%, only 17% asked their children to count or to state a cardinal value. The display box had a series of handled doors that could be pulled up to reveal something to count, for example, wheels on a bicycle, eggs in an egg carton, and so on. Although the children's attention usually was not drawn to the counting opportunities, this did not stop them from having fun. They treated the environment as a prop for banging, not for learning the count sequence. Of equal interest is the fact that the accompanying adults did not feel compelled to interpret the exhibit as intended by the designers. Otherwise, many more would have chosen to read the "How many?" questions on the display for their preliterate children.

Table 1. Examples of Ways Primary School Children Ordered Numerical Representations. (Each item was on a separate card and children knew they could self-correct and place more than one card at the same position).

Example Number/Salient Attributes
Example 1:
$0 \quad 1 \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad 1\frac{1}{2} \quad 1\frac{3}{4} \quad 2 \quad \frac{2}{2} \quad \frac{2}{4} \quad 2\frac{1}{2} \quad 2\frac{1}{3} \quad 2\frac{2}{3} \quad 3 \quad \frac{3}{3} \quad \frac{3}{4} \quad 3\frac{1}{2} \quad 4$
—0 & 1 before unit fractions with ascending denominator, then clusters of ascending whole numbers with "related" mixed fractions
—x/x not treated as equals
Example 2:
$0 \quad 1 \quad \frac{1}{2} \quad \frac{2}{2} \quad \frac{1}{3} \quad \frac{3}{3} \quad \frac{1}{4} \quad \frac{2}{4} \quad \frac{3}{4} \quad 1 \quad 1\frac{1}{2} \quad 1\frac{1}{4} \quad 1\frac{3}{4} \quad 2 \quad 2\frac{1}{2} \quad 2\frac{1}{3} \quad 2\frac{2}{3} \quad 3 \quad 3\frac{1}{2} \quad 4$
—0 before symbols that look like fractions; then whole number and "their fractions";
—ordering rule for fractions—rank order denominators and within each cluster rank the numerator
—x/x not treated as equals

There is a second implication of the view that learning depends on the existence of a mental structure that is at least incipiently available for interpreting the data. This is that, in the absence of the domain-relevant structure, the likelihood that learning about that domain will move forward is extremely low. As a result, we have the deep-seated problem of figuring out how one acquires new structures.

Simply providing data for practice does not result in the assembly of a structure. To move mathematical and scientific learning to a new level of conceptual understanding, you must create a new conceptual structure and build a domain-relevant knowledge base. The terms, examples, and key concepts that are part of the new target domain of learning are hard, if not impossible, to understand in the absence of a domain-relevant conceptual structure. This means that you have to solve the problem of teaching, in an integrated fashion, the content, language, notation, representational tools, and principles of that domain. The task is to piggyback the learning of all of these on each other to develop anything like understanding, let alone expertise.

To illustrate the interrelatedness problem, we begin with the way first-, second-, and third-grade children respond to a request to rank order a set of cards that include representations of zero, whole numbers, and what are called *mathematical wholes* (*X*/*X*). As is clear in Table 1, children of this age do not integrate natural and rational numbers. Their effort to rank order fractions appears to be based on the assumption that the symbols involving fractions are novel examples of natural numbers. Note especially that the children did not place the mathematical wholes where they put one—even though pretest training encouraged the idea that several items could go at the same place. This is consistent with another result from a suburban school system in the Midwest.

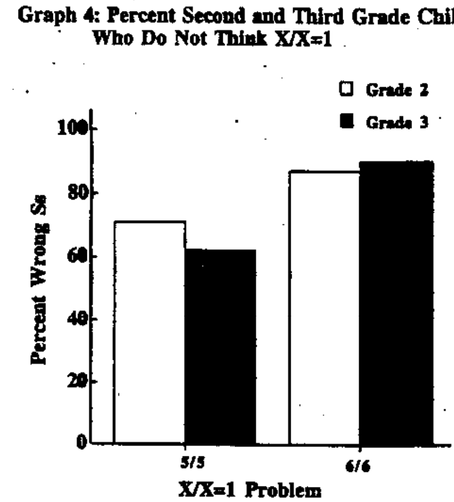

Figure 4. Children Do Not Think That $A/A = 1$.

As shown in Figure 4, even fourth- and fifth-grade children are *not* inclined to agree that 5/5 and 6/6 are equal to 1. Further, the correct answer as to whether 5/5 equals 1 does not guarantee the correct answer for 6/6. Hence, one does not want to attribute understanding of the underlying conceptual rule, $X/X = 1$ (no matter what value X assumes). We anticipated these results. Why? In part, the language of this mathematics does not have entries that are part of everyday discourse with numbers. When we "talk in English," we don't talk about one thing being the same as "five over five" or "six over six", and so forth. Additionally, it is one matter to answer examples that one has practiced (5/5) and quite another to achieve an induction of the mathematical interpretation of that instance. This consideration is related to what is both a methodologic and theoretic issue.

There are different ways that children can answer questions about fractional representations, some of which are not based on a mathematical understanding. They can match the symbol 1/4 with a circle in which one of its 4 equal parts is darkened by counting the number of darkened parts and the total number of parts. This would involve using a combination of a counting and perceptual matching strategy. They also can add and subtract parts of a circle by mentally "pushing" an "image" of the parts together. These solutions can lead to correct performance in rather young children (e.g., Mix, Levine, & Huttenlocher, 1999) even if they do not map onto the mathematical definition of a fraction, one number divided by another. Given this definition, it follows that the result can be either a natural *or* rational number, otherwise division would not be closed. It is for the latter reason that we say that the *mathematical* meaning of the representations of fractions, in either fractional or decimal format, depends on a principled mathematical reasoning.

From the psychological perspective, the idea is that understanding of rational numbers depends on a conceptual structure that is different from that used to reason about positive count numbers (Hartnett & Gelman, 1998). For example, although every count number has a successor, this is not true for rational numbers. In fact, there is an infinite number of numbers between any two rational numbers. When the problem is viewed from this perspective, it is not surprising that misunderstandings of rational numbers and their representations persist through high school and into the college years (e.g, Carpenter, Fennema, & Romberg, 1993; Fischbein, Deri, Nello, & Marino, 1985). I must admit however, I was taken aback when my colleagues in the chemistry department told me that a considerable number of their undergraduate students insist one can't graph a particular problem because 'there aren't enough squares on the graph paper.'

Conclusions

There is good evidence that learning about rational numbers is a difficult matter that begins early in human development. The growing evidence regarding the universal, domain-specific ability for children to learn and reason with numbers indicates that we are born with number-relevant mental structures that promote the development of principles for counting. The presence of these structures enables children to attend to and assimilate number-relevant information. The account of how young children self-monitor their number learning illustrates the argument with data about preschool children's organized understanding of natural numbers. Learners are active interpreters and they use whatever they already know to make sense of novel situations. Because of the latter, number learning is more than a matter of practice, a mental structure must be available to assist the learner interpret information, and simply providing data for practice does not result in the assembly of a structure. The college-age examples herein illustrate that it is a challenging task to create relevant learning paths for numerical understanding and that the problem is at least as much a developmental one as it is one of learning.

References

Carpenter, T. P., Fennema, E., & Romberg, T. A. (Eds.). (1993). *Rational numbers: An integration of research.* Hillsdale, NJ: Erlbaum.

Fischbein, E., Deri, M., Nello, M., & Marino, M. (1985). The role of implicit models in solving problems in multiplication and division. *Journal of Research in Mathematics Education, 16,* 3–17.

Gallistel, C. R., & Gelman, R. (1992). Preverbal and verbal counting and computation. *Cognition, 44,* 43–74.

Gelman, R. (1994). Constructivism and supporting environments. In D. Tirosh (Ed.), *Implicit and explicit knowledge: An educational approach* (pp. 55–82). Norwood, NJ: Ablex.

Gelman, R., & Cohen, M. (1988). Qualitative differences in the way Down Syndrome and normal children solve a novel counting problem. In L. Nadel (Ed.), *The psychobiology of Down Syndrome* (pp. 51–99). Cambridge, MA: MIT Press/Bradford Books.

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number.* Cambridge, MA: Harvard University Press.

Gelman, R., Massey, C., & McManus, M. (1991). Characterizing supporting environments for cognitive development: Lessons from children in a museum. In J. M. Levine, L. B. Resnick, & S. D. Teasley (Eds.), *Perspectives on socially shared cognition* (pp. 226–256). Washington, DC: American Psychological Association.

Gelman, R., & Meck, E. (1986). The notion of principle: the case of counting. In J. Hiebert (Ed.), *The relationship between procedural and conceptual competence* (pp. 29–57). Hillsdale, NJ: Erlbaum.

Gelman, R., Meck, E., & Merkin, S. (1986). Young children's numerical competence. *Cognitive Development, 1*, 1–29.

Gelman, R., & Williams, E. (1998). Enabling constraints for cognitive development and learning: Domain specificity and epigenesis. In W. Damon (Editor-in-Chief), & D. Kuhn & R. Siegler (Vol. Eds.), *Cognition, perception & language: Vol. 2. Handbook of Child Psychology* (5th ed., pp. 575–630). New York: Wiley.

Glaser, R. (1988). Cognitive science and education. *International Social Science Journal, 115*, 21–44.

Hartnett, P. M., & Gelman, R. (1998). Early understandings of numbers: Paths or barriers to the construction of new understandings? *Learning and Instruction: The Journal of the European Association for Research in Learning and Instruction, 8*(4), 341–374.

Mix, K. S., Levine, S. C., & Huttenlocher, J. (1999). Early fraction calculation ability. *Developmental Psychology, 35*, 164–174.