HANDBOOK OF MATHEMATICAL COGNITION

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The Young Numerical Mind

When Does It Count?

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Accounts of early counting differ in the degree of conceptual competence granted to the young child, as well as whether there are ontogenetic and/or phylogenetic continuities. Much of the debate is centered on whether various counting tasks license the conclusion that young learners understand the cardinal counting principle, that the last word in a count list represents the cardinal value of a collection.

Some theorists hold that a young child's initial count words have no numerical meaning for them. They first have to connect each of the first few count words to either a perceptual representation (Starkey & Cooper, 1995; Huttenlocher, Jordan, & Levin, 1994) or a nonverbal representation of the exact quantity for a given small N (Bloom, 2000; Wynn, 1990, 1992b). Others add that the language of count words grows out of the semantics of quantifiers in the language (Bloom 2000; Bloom & Wynn, 1997; Carey, 1998). Depending on the account, the requisite induction either co-occurs or sets the stage for learning the relationship between verbal counting and knowledge of addition and subtraction. Thus, it widely is assumed that learning the meaning of the counting procedure is developmentally prior to learning about addition and subtraction (but see Sophian, 1998). Gelman and Gallistel's (1978) account of a principled understanding of counting differs in this regard.

For Gelman and Gallistel, the counting principles always have been considered part and parcel of an implicit arithmetic structure, be it verbal or non-verbal. A meaningful verbal counting procedure is one that is consistent with the counting principles of: one-one (each item gets one and only one unique count tag), stable ordering (the count words are consistently used in a stable order), cardinality (the last word in the count represents the cardinality of the set), order irrelevance (the items may be counted in any order), and item-kind irrelevance (there are no restrictions on what counts as a countable entity). The execution of a competent plan of counting must be consistent with these principles for it to yield a cardinal value to which the operations of addition and subtraction can be applied. Thus, the counting principles do not stand alone. Successive count words represent ordered values because they are *subject to the axioms of arithmetic.* Below we argue that a domain-specific nonverbal counting and arithmetic structure provides very important domain-relevant clues for young children to use when learning the language and meaning of count words.

BACKGROUND

Much of the early work on numerical knowledge involves one or another counting task. Para digms in which assessments of counting are combined with its role in arithmetic are much les prevalent. This is not surprising if we take into account the theoretical position of many labs Given the assumption that young children do not understand their own counting, it hardl makes sense to ask them to relate counting to mathematical operations. For example, Piaget' (1952) theory is a set-theoretic one that grounds the understanding of cardinality in th operations of one-one correspondence and logical classification. For him, counting in pre operational children is done by rote and without understanding. Bermejo (1996; Bermejo & Lago, 1990) adopts the Piagetian position that preschoolers cannot understand the cardina principle on the basis of counting. This class of accounts rejects the view that preschooler have any numerical abilities.[1] Theories in the empiricist tradition (e.g., Baroody & Wilkins 1999; Mix, Levine, & Huttenlocher, 2002) share with developmentalists like Piaget the view that children must progress from the perceptual to the abstract—no matter what the conceptua domain. At first, children use only perceptual information to put together identical items fo a count. Then they move to classifying together items of the same shape but of different colo then items that differ in kind but share color, and so on, until they can collect for a coun widely heterogeneous "things." Understanding that a final count number, say 5, represents an set of 5 requires the use of an abstract classification structure. This criterion converges wit Piaget's regarding the development of classification.

Mix et al. (2002) also place considerable emphasis on the role of language. McLeish (1991 shares their overall perspective. "The reason for animals' (and preverbal children's [authors addition]) inability to separate numbers from the concrete situation is that they are unable t think in the abstract at all—and even if they could, they have no language capable of commun cating, or absorbing, such abstract ideas as 'six,' or a 'herd'" (p. 7). Carey's (2001b) language dependent account grants some role to a system for nonverbal counting and/or arithmeti But for her, the verbal counting system reflects a conceptual change that is closely related t and emergent from the semantic/syntactic linguistic system of quantifiers (also see Spelke 2000, for a related account).

We take up the differences in theoretical perspectives by considering two interrelated topics (1) the evidence for early counting abilities and (2) the possibility that all humans, includin preverbal infants and toddlers, possess an implicit understanding of the relationship betwee counting and the arithmetic principles.

MATTERS OF EVIDENCE

The Counting Tasks

One popular counting task, the "How Many?" one (HM), involves showing children N object and asking "How many [objects]?" Task variables have included homogeneous vs. heterogeneou items, 2- vs. 3-dimensional pictures or objects, events or sounds, explicitness of instruction prior to the HM question, and so on. Across these conditions, the data converge. First, majority of children succeed on at least some variant of the task. The older the child, th greater the probability of a correct count within and across conditions (Fuson, 1988). Second the probability of the child repeating the last count word in response to an HM question is als a function of development. Third, the younger the child, the more likely it is she will be ope to misinterpretations of what to count and of variables that influence the production of one one and tag-generation errors. As such, younger children benefit from being told to coun more slowly, touching and moving the items, and smaller set sizes (Gelman & Tucker, 1975) There is debate about the interpretation of these variables as well as scoring criteria. Som treat a child's emphasis of the last word in the count as positive evidence (Gelman & Gallistel, 1978). Others insist that even repetition of the last count word reflects nothing more than the child's imitation of a social rule of counting (Fuson, 1988). The difficulty of interpreting data from the seemingly straightforward HM task led us to work up alternative tasks.

Gelman, Meck, & Merkin (1986) designed a paradigm to test young children's flexibility with aspects of the counting procedure. In their study, 3-, 4-, and 5-year-olds watched a puppet count an array of objects and were told, "It is your job to tell [puppet] if it was OK to count the way he did or not." Children in this study were highly sensitive to violations of the one-one and cardinal principles, correcting the puppet when he double-counted, skipped an item, or repeated an incorrect cardinal value. In a different study–the "Doesn't Matter" task (Gelman & Gallistel, 1978; Gelman, Meck, & Merkin, 1986; Gelman & Meck, 1983, 1986)–preschoolers were asked to count a row of objects in an unconventional manner, by making an item in the middle be the "one" or the "two" in the count. Children quickly adopted successful strategies for complying, while honoring the counting principles. Challenges based on the order-irrelevance and item-irrelevance tasks appeared from various labs (Baroody & Wilkins, 1999; Briars & Siegler, 1984). However, Cowan, Dowker, Christakis, & Bailey's (1996) subsequent studies showed that 3- to 6-year-old children did well at working with novel counting examples if they did not have to be metacognitive. The authors concluded that question format is likely to influence whether children attend to underlying principles or various performance variables. This fits Gelman and Meck's (1986) report that youngsters benefit from instructions about the difference between a "silly" and "wrong" count sequence. But there must be more to the story.

Children younger than 3½ years almost always fail Wynn's "Give-N" task (1990, 1992b), which asks a child to give a puppet one to six small animals. Developmentally, youngsters can give one item before they give two items, and two before three animals. Wynn reports that once a child produces four items (at about 3½ years), she also does so with all larger set sizes in her count list. Until then a child engages in "grabbing." Wynn concludes that the shift reflects an understanding of the cardinal principle that "helps them to immediately acquire the meanings of all the number words [in their count list]" (Wynn, 1990, p. 186).

Although these results are robust, there is reason to question whether the "Give-N" task is a fair test of the acquisition of verbal counting. Brannon & Van de Walle (2001) observed different behavioral responses to the "Give-N," as opposed to the HM and "What's on the Card" (see below) tasks. Although children were quick to respond in the latter two tasks, they hesitated and verbalized much more confusion when participating in the "Give-N" task. Our lab made similar observations in a replication of the task. We conclude that the "Give-N" task is a very hard counting task, possibly for the following reasons.

In the "Give-N" task, the child has to create a set of objects, one by one, until she has created a set whose numerical value corresponds to one in memory. These conceptual requirements overlap to a considerable degree with variants of the number conservation task that have been paired with counting (e.g., Becker, 1989; Fuson, 1988; Gelman, 1982). Results of these task variations all agree that it is far from easy to get preschool children to use cardinal count values to construct and/or compare two sets. Becker (1989) reported that only some 3½-year-old children used the cardinal value resulting from counting each of two sets in correspondence to decide whether these were equal or not on the grounds of one-to-one correspondence. The convergence of ages in the "Give-N" and Becker tasks makes sense. Both involve using cardinal values in memory to generate or compare an equivalent value. The combined competence requirements exceed those of a beginning language user (Halford, in press).

No matter what the task, there is another consideration when beginning speakers are the subjects. Their language is very limited. The risk is nontrivial that whatever the instructions, they could be misinterpreted. To deal with this, we developed the "What's on the Card?" (WOC) task (Gelman, 1993). Subjects were three groups of children between the ages of 2½ and 3½ years, all within the age range who fail the "Give-N" task. The stimuli were a number of sets of cards, each with 1 through 7 stickers of given kind. As expected, the WOC question elicited a label response on the first card in a set, e.g., "a bee." The experimenter then said "that's right, that's a one-bee card" or "there's one bee," to communicate that the task was a number and not a labeling one. The subtle manipulation worked. Both 3-year-old groups performed at or above 80% for set sizes up to 6. The 2½-year-old group also performed better than predicted by Wynn's theory, especially when the score was based on all correct trials, including those with the minimal prompt of some pointing. Seventy percent of the youngest children counted and stated the cardinal value on set sizes up to 4.

Bullock & Gelman's (1977) instructions for their magic show avoided any talk about numbers. Still, their 2½-year-old subjects transferred an initial ordering relationship between 1 and 2 items to the novel displays with 3 and 4 items. Gelman (1993) analyzed the kinds of things that 2½-year-olds said following their encounter with the transfer displays. More than 60% of the 2½-year-olds either spontaneously counted the small sets or responded differentially to "How many . . ." and "Can you count . . ." questions, thus demonstrating the ability to relate knowledge of the cardinal principle to the ordering relation embedded in the count list. The counting observed here was in the name of explaining or thinking out loud about the relation between two pairs of ordered cardinal values. This could not occur if the children were not able to make arithmetic judgments to start. Nevertheless, even here the counting was variable. The same is true for the WOC task. This does not surprise us. A competent plan of action and its successful output involves much more than the use of the constraints of implicit knowledge of the counting principles and their representation of numerosity.

In addition to the issues considered above, there is one extremely demanding skill necessary for success on counting tasks, mastery of the count list. Once children identify the string of sounds that are relevant, they have to memorize a long list of words that lacks inherent structure. There is nothing about each of the sounds to indicate which one will follow. Children also need to cope with the information-processing demands involved in a successful output. For example, they have to coordinate the drawing of tags with points and separate counted from to-be-counted items. This is just the beginning of a discussion of the demands on performance (Gelman & Greeno, 1989; Canobi, Reeve, & Pattison, 2003). It should be clear why analyses of response variability should be the rule: even if children actively assimilate performance examples of counting to their nonverbal understanding, this process takes time and practice.

Arithmetic Tasks: A Window to Counting Competence

When knowledge of the effects of addition or subtraction is assessed with a magic show (in which items are surreptitiously added or removed from a set), many 2½-year-olds notice the change in the number of objects for small sets (Gelman, 1972). Sophian and Adams (1987) showed that even toddlers exhibit sensitivity to effects of arithmetic transformations on small sets. Hughes (1981) also found that 3-, 4-, and 5-year-old children were successful in solving simple addition and subtraction problems involving sets as large as 8. Zur and Gelman (2004) found similar results when they asked 3-, 4-, and 5-year-olds to make predictions about changes to a set. Each block of problems in their study started with a child counting a given number of items, e.g., donuts in a donut shop. Then she heard about the delivery or sale of N (1, 2, or 3) donuts. Next she predicted, without counting, how many items there were. Finally, she counted to check her prediction. Ninety percent of predictions were in the right direction, even for the youngest subjects, with a large proportion of responses differing from the correct value only by +/−1. The youngest children's responses (3:1–3:5 years) did not differ significantly from those of the older children, suggesting even the youngest subjects had some understanding of the cardinal principle.

In sum, when very young children count in the name of an arithmetic goal, data suggest that they do understand the verbal cardinal principle. We consider it premature to rule out the possibility that young children's learning of verbal counting benefits from a nonverbal counting and arithmetic.

NONVERBAL COUNTING DATA

Animals and Humans

A number of studies reveal that infants discriminate sets based upon the number of items in each set (e.g., Antell & Keating, 1983) and that they are even sensitive to set manipulations involving simple arithmetic (Wynn, 1992a). Numerical estimation data from rats, pigeons, monkeys, and humans of all ages yield similar data (Brannon, Wusthoff, Gallistel, & Gibbon, 2001; Mechner, 1958; Platt & Johnson, 1971; Rilling & McDiarmid, 1965), Pet dogs and dolphins can pick the larger of two sets (Kilian, Yaman, von Fersen, Gunturkun, 2003; West and Young, 2002), and even salamanders appear sensitive to the numerosity of stimuli (Uller, Jaeger, Guidry, & Martin, 2003). It is hard to continue to maintain that linguistic capacity is a condition for the representation of approximate quantities. The comparative data, in combination with those from adult psychophysical studies, open the door for our position that the nonverbal system serves as a foundation upon which the human verbal/symbolic numerical system is built.

Regardless of the species and task involved, a similar pattern of responses almost always is obtained. They obey Weber's law: that is, the "just noticeable difference" between two values is a constant proportion. More commonly, the ease (i.e., speed and accuracy) with which two numbers are discriminated is dependent upon the ratio of the two values (not their arithmetic difference, as one might suspect). This Weber characteristic is evidenced in the scalar variability found in the behavioral data, such that the variability of responses increases in proportion to the mean response. More precisely, the ratio of the standard deviation to the mean (coefficient of variation) is a constant value.

Scalar variability is not unique to the animal counting data. A constant Weber fraction has been measured in humans for a wide variety of perceived magnitudes, including weight, temperature, surface roughness, and duration (Stevens, 1970). Psychophysical research suggests that this is also the case with animals, as the extensive literature on animal timing reveals scalar variability to be a robust finding in the behavioral data. These cross-species and cross-modality consistencies suggest a similar mechanism of nonverbal representation for all quantities, both continuous and discrete (see Gallistel and Gelman, 2000; Walsh, 2003). Recently, direct tests of nonverbal counting abilities have revealed that humans share with animals this ability to represent approximate numerical values with scalar variability (Barth, Kanwisher, & Spelke, 2003; Cordes, Gelman, Gallistel, & Whalen, 2001; Whalen, Gallistel, and Gelman, 1999; with children, Huntley-Fenner, 2001; Whalen, Gelman, Cordes, & Gallistel, 2000).

These nonverbal counting results are not all that surprising. It is known that numerical discriminations in both animals and humans obey Weber's law, such that the speed and accuracy with which two sets are discriminated is negatively correlated with the absolute size of the sets and the numerical difference between the two sets. These numerical size and distance effects, respectively, have been demonstrated in animals, adult humans, and preschoolers (e.g., Dehaene & Akhavein, 1995; Huntley-Fenner & Cannon, 2000; Meck & Church, 1983). More interesting, however, is that the Weber characteristic of numerical discriminations holds even when the sets to be discriminated are replaced with Arabic numerals, suggesting that, at least in numerically fluent individuals (human or primate), the meanings of symbolic representations of numerosity are closely related to approximate nonverbal representations (Moyer & Landauer, 1967, 1973; Washburn, 1994).

Overall, we now know that humans and animals share a nonverbal counting ability to generate approximate representations of the Ns used in various tasks. For us, this implies that human infants similarly can approximate numerical representations that obey Weber's law.

Quantification in Infancy

A variety of habituation and preferential looking studies suggest infants are sensitive to displays of some numerosities (although see Mix, Huttenlocher, & Levin, 2002). For example, when habituated to a display of two objects, infants will then look longer to subsequent displays of three as opposed to two, and vice versa (Antell & Keating, 1983; Starkey & Cooper, 1980; Strauss & Curtis, 1981). Five-month-olds are sensitive to arithmetic transformations of small sets (Wynn, 1992) and changes in the number of grouped sets of moving dots (Wynn, Bloom, & Chiang, 2002).

The majority of work with infants has used visual arrays in which all items to be enumerated are presented simultaneously. A handful of studies have used sets of sequential events or sounds. For example, 6-month-olds discriminate between 2 and 3 jumps of a rabbit puppet (Wynn, 1996), and 4-day-old newborns' sucking rate habituates to a 2 (or 3) three-syllable utterance, they recover when they hear a 3 (or 2) syllable utterance, and vice versa (Bijeljac-Babic, Bertoncini, & Mehler, 1993). Infants preferred to look at a display with the same number of household objects as sounds they heard (Starkey, Spelke, & Gelman, 1983). Similarly, when a causal relationship was established between dropping objects and noises, 6-month-old infants expected a cross-modal match in numerosity (Kobayashi, Hiraki, & Hawegawa, 2002) (but see Mix, Levine, & Huttenlocher, 1997; Moore, Benenson, Reznick, Peterson, & Kagan, 1987[2]).

Preverbal numerosity discrimination is not limited to sets < 4. Xu and Spelke's (2000) 6-month-olds detected changes in the numerosity of visual arrays for large sets as long as the ratio of the numerosities was 2:1 (e.g., 16 from 8, and 32 from 16), but not when the ratio was as small as 3:2 (e.g., 16 from 12, or 32 from 24). A comparable result holds for sequentially presented sets (sounds; Lipton & Spelke, 2003).

Evidence from these studies and others lead us to favor the proposal of a nonverbal quantification system in humans. Many, but not all (Simon, 1997, 1999), concur. Still, there is debate about which kind of quantity-relevant representational system(s) best accounts for the data. Issues regarding discrete vs. continuous and numerical vs. non-numerical representations take center stage in the debate about the processes involved in infant quantification. Demonstrations of set-size discrimination limits around three or four items contribute heavily to differences between accounts. A key question is whether the quantification abilities of infants reflect the same nonverbal system revealed in the adult nonverbal counting tasks.

NONVERBAL QUANTIFICATION: HOW DO THEY DO IT?

The Accumulator Model

Originally proposed by Meck & Church (1983) to explain both the timing and counting data from nonverbal animals, the accumulator model has been adopted to explain nonverbal quantification abilities in the human domain as well (Gallistel & Gelman, 1992; 2000). According to this model, objective quantities (e.g., time, number, distance) are represented subjectively as continuous magnitudes in a mental accumulator. The mapping between continuous objective values (time, distance, intensity, amount, etc.) and continuous subjective values is a straightforward one--a small objective amount equates to a small subjective magnitude and a large amount equates to a large magnitude, such that there is a simple linear relationship between the two (but see Dehaene, Dupoux, & Mehler, 1990, for an alternative logarithmic account). But what about the case of number in which discrete objective values are mapped to continuous subjective magnitudes?

The case of number is a special one. The sole distinction between representations of discrete number and continuous values is found in the process by which mental magnitudes are mentally accumulated. While the accumulation process for continuous values is continuous (likened to running water from a hose into a bucket), nonverbal counting produces continuous magnitude representations via a discrete process. Each enumerated item increments the magnitude in the mental accumulator by an equal amount (equivalent to a cup of liquid being poured into the bucket for each enumerated item). Thus, whereas continuous and discrete objective values are represented by similar continuous mental magnitudes, the process by which these magnitudes are accumulated is distinct. The nonverbal counting process obeys the basic counting principles described earlier—each enumerated item is represented by one cup of activation (one-one) and each cup of activation increases the accumulated magnitude in memory, such that there is a discrete "next" (stable order), and the resulting magnitude represents the cardinality of the set (cardinality).

These mental magnitudes are not precise representations, however; that is, they are more appropriately described as probability density functions than as absolute values. Following the accumulation process, the resulting magnitude is transferred to memory, where it is subject to scalar noise—noise proportional to the magnitude of the representation. Thus, the scalar variability observed in the behavioral data is a reflection of the inherent scalar variability in the system.

The Mapping between Nonverbal and Written Number

The accumulator model provides a satisfying account of the ubiquitous data, indicating that magnitude discriminations obey Weber's law. If objective magnitudes are subjectively represented via a magnitude system with scalar variability, the ratio of two values directly reflects the extent to which two values (probability density functions) overlap; thus, as the ratio of two values approaches one, the amount of overlap in the two representations increases, making the values subjectively more similar. The fact that symbolic representations of numerosity obey Weber's law strongly suggests that at some point in the development of numerical fluency, a bidirectional map is achieved between the numerical words/symbols and magnitudes represented in the accumulator system. The question is when does this happen.

A number of recent studies with children have used a numerical Stroop-like task (in which numerical and physical size of the stimuli contrast with one another) to show the effect of schooling. Rubinsten, Henik, Berger, & Shahar-Shalev (2002) found that when children in the beginning of first grade chose the greater of two quantities, number did not interfere with their success. At the end of their school year, the children's quantity judgments were negatively influenced by the presence of irrelevant number dimensions. By the end of the same grade, there was interference.

An unpublished study by the authors and John Whalen in our UCLA lab[3] complements these kinds of results. Kindergarten through fifth-grade students and adults indicated which of two Arabic numerals (2–9) was larger, either numerically or physically. The stimuli differed in both physical and numerical size such that the numerically larger digit was either physically larger (congruent) or physically smaller (incongruent) than the other digit. Repeated measures analyses of variance of the median times[4] ($p < .05$, at least) revealed a reliable advantage of physical size, such that physical size of the display items interfered with judgments of numerical value for all age groups. Numerical size interfered with judgments of physical size for all age groups except for the youngest subjects (< 6½ years). Not only did the numerals fail to compromise the youngest children's response times when judging incongruent stimuli, they did not elicit a distance effect in the numerical condition. These two effects lead us to conclude that our youngest subjects' reading of numerals is very slow. Still, the process seems to be well on the way with about a year of schooling, which suggests to us that its beginnings should be placed earlier in development. Nöel, Rousselle, & Mussolin (chapter 11) provide relevant evidence in their extended treatment of the mapping between culturally defined symbols and nonverbal representations of numerosity practice.

Infant Accumulators

The animal and (noninfant) human nonverbal counting data are described well by an accumulator system. However, do magnitude representations of quantity explain the pattern of results obtained in the infant counting data? Clearly, some of the evidence for preverbal quantification can be accounted for by an accumulator-like system. Since the system is truly numerical by nature, there are no restrictions on what sorts of sets can be enumerated. Thus, data suggesting infants enumerate spatial displays as well as sequential events and match numerosities cross-modally are expected. In addition, there is no clear upper limit on the possible cardinal values this system can handle, so evidence of infant discrimination of sets as large as 32 from sets of 16 (and 16 from 8) also support claims of magnitude representation in these young subjects.

The Weber law characteristic of magnitude representations may be challenged, however, by some of the infant data. Infants successfully discriminate sets with ratios of 2:1, whereas they fail on tests of ratios of 3:2 in the "large number range" (16 vs. 12 or 32 vs. 24; Xu & Spelke, 2000). Under a magnitude-representation account, this result would suggest that the Weber fraction for a just-noticeable difference in number for infants this young was closer to .3 log units (2:1) than to .18 log units (3:2); thus, sets with logarithmic differences closer to .18 may simply be beyond the infants' discrimination capabilities. This account is questioned, however, by a variety of data revealing infants are able to successfully discriminate sets of 3 from 2 (e.g., Antell & Keating, 1983). These results suggest that the Weber fraction for numerosity discrimination may actually decrease in the small number range for infants.

The lack of a consistent Weber fraction for numerical discrimination in infants threatens theories positing magnitude representations as the sole basis for infant quantification results. The accumulator model incorporates scalar variability in order to account for the Weber characteristic of magnitude discrimination. The model does not make special allowances for violations of Weber's law. The infant data, however, suggest that numerical discrimination is subject to a Weber fraction of around .3 log units (2:1 ratio) but only for values *greater* than 3 or 4. For values under 4, discrimination abilities appear to be keener than predicted by this fraction. Why?

Perhaps the accumulator system is used only to represent values greater than 4. This cannot be the general case. Scalar variability characterizes the adult psychophysical function in both the small and large number ranges, without any signs of discontinuities below values of 3 or 4 (Balakrishnan & Ashby, 1992; Cordes, Gelman, Gallistel, & Whalen, 2001). In addition, animals are known to treat values above and below 4 in similar ways (Brannon & Terrace, 1998, 2000; Meck & Church, 1983).

Differences in data collection procedures for infants may contribute to their different response pattern. The majority of infant studies employ either preferential looking or habituation paradigms with amount of looking time as the crucial dependent variable. This implicit measure contrasts with the explicit responses obtained in both animal and adult human nonverbal counting tasks (i.e., lever/button presses or pecks). So, too, does the fact that most infant analyses are based on group data. Information regarding individual variability, patterns of responses, and developmental levels (necessary for arguments regarding underlying representations) are simply not available. Still, the apparent inconsistent Weber fraction for infants cannot be ignored as the results are fairly robust. Although the accumulator model accounts for most of the infant quantification data, the change in the Weber fraction for values fewer than four needs explanation. Many have proposed that the small number range is processed with object files/indexes.

Object Files

Kahneman, Treisman, & Gibbs (1992) introduced the notion of object files (or object indexes) to describe how humans track objects in their visual environment. Object files are often referred to as mental pointers with a record of some minimal information about the object being tracked, such as its location and shape. This information is used to identify objects, allowing one to determine if an object retrieved from behind an occluder is the same or different from the one originally placed behind the occluder (i.e., object permanence).

Although object files are discrete and can be counted, they are not numerical representations. They are merely mental pointers, and any estimate of the numerosity of the set of tracked objects must be performed via some other representational system (e.g., the accumulator system). A number of studies on multiple-object tracking by Pylyshyn and colleagues (1989; Trick & Pylyshyn, 1994; Scholl & Pylyshyn, 1999) have determined that there is also a limit on the number of objects one can simultaneously track. In adults, this limit is about 4 or 5, although with repeated practice, certain individuals have been able to track as many as 9 objects. Individuation experiments with infants suggests that this limit may also be subject to developmental changes, as work indicates that infants may only be able to track as many as 3 or 4 objects (Leslie, Xu, Tremoulet, & Scholl, 1998).

Object files have been implicated as an alternative representational system used for small set discrimination in both infants and adults (e.g., Leslie et al., 1998; Trick & Pylyshyn, 1994). Because object files are discrete and precise (noise free) by nature, they provide a viable explanation for the fact that studies have found that infants are able to discriminate 2 objects from 3 objects, but not 4 from 6, despite a similar ratio. The idea is that an accumulator system is used primarily for large number representations (wherever discrimination is ratio-dependent) and object files are used for small numbers (where discrimination has a set-size limit).

There have been claims that object-file representations "underlie most, if not all, of the infant successes in experiments that involve small sets of objects" (Carey, 2001a, p. 313). Although object files provide an account for much of the infant discrimination data for sets smaller than 4, they cannot possibly explain it all. Object files are constructs of the visual attention system, used for identification and mental tracking of discrete visual objects. By virtue of this definition, studies indicating infant discrimination of sounds or rabbit hops (e.g., Lipton & Spelke, 2003; Wynn, 1996) fail to be accounted for by object files, because events are not visual objects. Since these studies reveal successful discrimination of 3 events from 2 (Weber fraction of .18 log units), these data cannot be explained by an accumulator system of representation, either. Clearly, limitations on both of these models prevent a full explanation of the infant data. Further investigation into infant quantification abilities (preferably using repeated trials) as well as modifications to current theories or the introduction of novel ones are necessary for a greater understanding of the basis for numerical competence.

Continuous Extent Representations

There have been a number of developmental studies indicating that infants are able to differentiate sets based upon the quantity of continuous variables such as overall surface area, perimeter, or volume (independent of number). Recently, Feigensen, Carey, & Hauser (2002) placed different numbers of graham crackers of varying sizes into two different buckets in front of 10- and 12-month-old infants. Following the placement of the crackers, the infants were allowed to crawl to one of the buckets in order to retrieve the graham crackers. Results revealed that infants in their study crawled to the bucket containing the greater overall amount of crackers, even if that bucket contained the fewer (cardinal) number of crackers (interestingly, however, once either bucket contained more than four pieces of cracker, responding decreased to chance levels–suggesting object files also played a role in this task). Thus, it appears that infants also use a measure of continuous extent (i.e., amount of cracker) as a relevant dimension in set discrimination.

Because of results such as the ones reported above, most researchers involved in early quantification work agree that the data point to the existence of preverbal representations of both numerosity and continuous stimulus magnitudes. There are others, though, who offer an alternative, and decidedly different, explanation of the data. Mix et al. (2002) contest the proposition that infants are capable of evaluating sets based upon discrete numerosity. They insist that the successes found in counting experiments all are based upon representations of continuous stimulus properties. They argue that "development starts with only one principle of quantification in infancy based on amount of substance, which applies to both continuous quantity and sets of discrete objects" (p. 62). According to their account, "infants do not represent quantities numerically at all. Instead, the evidence points to the use of overall amount" (p. 81).

Principal support for their theory stems from failures to find evidence for infant sensitivity to numerosity once continuous stimulus variables are controlled. For example, in Clearfield & Mix (1999), 6- to 8-month-old infants were habituated to displays of either 2 or 3 objects. Following habituation, the infants saw two test displays: (1) displays with the same numerosity but different overall contour than the habituation displays (extent test) and (2) displays with a novel number of items (3 or 2) but with the same amount of overall contour (number test). Subjects looked significantly longer than during habituation at the extent test displays but did not behave as if they noticed a change in the number test displays, suggesting their task tapped into representations of continuous extent, not numerosity.

Although it is likely that some infant counting results are due to distinctions of continuous extent (and not number), it is unlikely that this is ubiquitously so. There are a number of studies that fail to be explained by infant quantification of continuous extent alone. Xu and Spelke (2000) varied continuous extent while maintaining number constant throughout the familiarization trials of their large number discrimination tasks. If the infants were only sensitive to changes in continuous extent, it would not be possible to habituate them to these stimuli. They did. In a study of ordinal relations by Brannon (2002), infants were habituated to sets of displays of increasing or decreasing numerosity (2-4-8 or 8-4-2) while controlling for overall surface area of the stimuli. When presented test displays of either ascending or descending numerosities (also with the same surface area), the 11-month-olds dishabituated to the novel ordering, suggesting they had attended to the numerical ordering of the habituation displays, not the extent. Most recently, Leslie, Glanville, & Lerner (2003) and Brannon & Gautier (2003) pitted continuous extent against number and found infants in their tasks responded significantly more to changes in number.

These projects, as well as studies of event number discrimination (e.g., puppet jumps, cross-modal matching, sound enumeration—Wynn, 1996; Starkey, Spelke, & Gelman, 1980; Lipton & Spelke, 2003) in which it is entirely unclear what would be defined as the continuous extent variable, imply that infants must also be sensitive to numerosity. For the reasons presented above, we can conclude that any representational model strictly confined to quantification of continuous magnitudes is limited and insufficient for explaining the data.

Infants and Quantities: What's the Story?

In sum, work on infant quantification suggests that preverbal infants are sensitive to changes in numerosity as well as to changes in continuous amounts (surface area, perimeter). Certain data sets revealing infant discrimination of continuous extent (Clearfield & Mix, 1994), of large numerosities (Xu & Spelke, 2000), and/or of the number of event sequences (Lipton & Spelke, 2003) can only be accounted for by an accumulator-like representational system. However, indications of a shift in the Weber fraction for values smaller than 3 also implicate object-files in some tasks involving small numerical values, provided the stimuli are discrete, visual objects. We propose that the infant successes in quantification tasks are due to an interaction of object file representations and approximate magnitude representations of both number and continuous variables. It is suggested that a number of task variables influence which mechanisms (object files or accumulator) and which relevant dimensions (number, surface area, etc.) present themselves in the data.

SOME CRITICAL POINTS

The persistence of the arguments from both the skill-first and language-dependency camps has taken the turn of listing the reasons why the accumulator model for animal counting cannot apply to the human case. The force with which they are presented is beginning to legitimize them, even though the arguments are faulty. We take up the statements to this effect one by one:

1. *The sequential nature of the nonverbal counting process makes enumeration of static sets impossible or at least more difficult than enumeration of sequential sets. "Seemingly, it would be difficult for infants to keep track of which items in a visual set had been 'accumulated' without physically partitioning the set, as we do in verbal counting" (Mix et al., 2002, p. 90). This comment is also relevant to the adult data, as new evidence suggests that the time required to discriminate large sets is solely a function of the ratio of the two sets, NOT their overall magnitude (Barth, Kanwisher, & Spelke, 2001).*[5]

The underlying accumulator system is not necessarily sequential by nature. The physical model of the accumulator is sequential in order to simplify understanding of the mathematical model. The true mathematical model of the system does not require each "cupful of activation" to be poured one after the other. It is quite possible to imagine that the cups are poured in all at once, or perhaps there is a limit to the number of cups poured at once (the same as the object file limit?). As Brannon points out, "It may be that two distinct processes yielded large approximate number representations; an iterative counting like procedure operating over sequentially presented arrays and a parallel mechanism operating over simultaneous arrays" (2003, p. 281).

The accumulator model is not strictly committed to an iterative nonverbal counting process and equally accounts for the enumeration of sequential and static sets. It should also be noted that the nonverbal counting routine is naturally implicit. It is not a conscious process–it occurs as one (whether in an infant, adult, or nonhuman animal) scans the display and does not require conscious partitions of counted items vs. to be counted items.

2. *The accumulator is only used for representations of number or duration, and not other continuous variables such as surface area or contour.*

The accumulator model goes hand in hand with the concept of mental magnitudes. These magnitudes are used to describe the subjective representation of all objective quantities that obey Weber's law. Although the accumulator model was originally proposed to account for the animal counting and timing data, more generally, it has been adopted to explain all subjective magnitude representations, including surface area, density, and length.

3. *Evidence of nonverbal arithmetic in infants (e.g., Wynn, 1992a) "cannot be explained without positing complicated maneuvers involving multiple accumulators" (Mix et al., 2002, p. 91).*

Central to the Gallistel & Gelman (1992) account of preverbal counting, subjective magnitudes are in the service of the arithmetic principles. Studies with both humans and animals reveal that magnitudes are subject to nonverbal computations (in animals–Boysen & Berntson, 1989; Brannon, Wusthoff, Gallistel, & Gibbon, 2001; Gibbon & Church, 1971; Leon & Gallistel, 1998; in adult humans–Barth, 2001; Cordes, Gallistel, Gelman & Latham, 2004; Zacks & Hasher, 2002; in infants–Aslin, Saffran, & Newport, 1999; for reviews, see Gallistel, 1990; Gallistel, Gelman, & Cordes, in press). While physical instantiations of how this works may be complicated, evidence suggests that arithmetic manipulations of accumulated magnitudes are regularly performed online.[6]

4. "There is no direct evidence for the accumulator mechanism in infants and children" (Mix et al., 2002, p. 91).

Nonverbal counting studies with young children have revealed the same scalar variability signature found in nonhuman animals (e.g., Huntley-Fenner, 2001; Huntley-Fenner & Cannon, 2000; Whalen, Gelman, Cordes, & Gallistel, 2000). There is also plenty of indirect evidence of magnitude representations of numerosity in infants as well as evidence of sensitivity to ordinal relations (Brannon, 2002).

5. Limitations on the magnitude values in the accumulator are not mirrored in the generative verbal count list (Carey, 2001a). While the natural numbers proceed to infinity, the list of magnitude values represented is finite.

This has been a criticism of the accumulator model, provided a linear mapping between objective and subjective magnitudes.[7] How do we deal with excessively large values without stressing the bounds of working memory? We propose an account similar to the "relative amount" case suggested by Mix et al.–the magnitude of the values in the accumulator is relative, not absolute. The magnitude of the representation of a given value varies as a function of the magnitude of the other values currently in working memory. For example, when dealing with numerical values of 1–50, the magnitude for 10 may look like this: ________. However, when working with values 1–100, the magnitude for 10 may only be: ____.[8] Magnitude sizes are determined by anchor values. This relative magnitude account allows for subjective representations of significantly large values without exceeding the capacities of working memory.

6. Nonverbal representations are inherently different from the verbal ones, as are the two counting processes.

Both routines strictly adhere to the basic how-to count principles, and both representational systems are ordered and embody a discrete process for generating new counts. Children may take time to learn the mapping between the two systems, but this is a function of the time it takes to memorize an ordered list, implicitly determine the parameters involved in the mapping between verbal and nonverbal magnitudes (see Cordes et al., 2001), and learn how to apply these nonverbal principles to the verbal domain. Noise in the nonverbal representation system also extends the acquisition process.

7. Subitizing is a real phenomenon.

Subitizing is reported as the rapid apprehension and identification of the numerosities of small sets (1–4) from a visual scene without counting. The literature on subitizing is closely related to that of object files (Trick & Pylyshyn, 1994). It could be said that subitizing is the rapid enumeration of open object files, without counting (verbal or nonverbal).

There is little clear evidence of subitizing in infants or young children. The evidence of this ability in adults is also questionable (Gallistel & Gelman, 1991). Mandler & Shebo (1982) found the observed subitizing data to be a function of recognizable canonical patterns (e.g., two points make a line; three, a triangle . . .). Balakrishnan & Ashby (1992) reanalyzed reaction time data from a variety of studies claiming to provide evidence of this ability in adults. Researchers who originally obtained the RT data had claimed the slope of the RT function in the small number range was significantly shallower than the slope of the RT function past that range (thus producing an "elbow" of discontinuity in the curve). The rigorous statistical tests run by Balakrishnan and Ashby failed to support this claim, and they concluded that subitizing was not, in fact, a true phenomenon. Whalen, West, & Cook (2003) also recently compared both response times and errors obtained when adult subjects were asked to count vs. estimate the numerosity of a set (size 1–16). Analyses revealed that what previously has been cited as evidence of subitizing (a shallow slope in the small number range) was well described as the result of nonverbal counting in a range where there is very little noise in the representation. Last, variability analyses revealed that there is no evidence of subitizing in the case of sequential stimuli (Cordes et al., 2001; Gelman & Cordes, 2001).

SUMMARY

Currently, the preschool counting data fail to provide conclusive evidence regarding competence of the early counter. While many studies point to a principles-before account, in which children have an inherent understanding of the basic counting principles, there are results from other studies that do not merit such strong conclusions. It is our position that even the data from these studies, in which children do not consistently act in accordance with the cardinality principle, support the existence of innate skeletal principles. Numerous factors, including strenuous task demands that tap into performance variables (as opposed to conceptual competence) and the expected trial-and-error associated with the acquisition of all skills, necessarily contribute to the observed variability in the data.

The available counting tasks are limited in their scope and breadth of assessment. There is a real need for new experimental paradigms to further investigate these issues and to provide more conclusive evidence. These tasks must be designed for use with children just beginning to count in order to look at the abilities of subjects younger than 3½ years old. As it is uncertain whether or not young children understand what the cardinal value of a count reveals, we favor investigations of early arithmetic competence, such as those demonstrated by Zur and Gelman (2004), as a means of determining counting competence.

The nonverbal counting data may also provide insight into the mind of young counters. Data from numerous studies support the existence of preverbal representational mechanisms for both number and continuous dimensions such as time, surface area, and contour. We propose that these representations are best described as accumulator magnitudes and object files, both of which appear to be available to the infant. Provided that object files are non-numerical by nature, we further suggest that it is the magnitude representations that are responsible for the young child's understanding of number and arithmetic. This nonverbal system provides the framework for the child to acquire a verbal count routine. The bidirectional mapping between this system and the linguistic one also allows the child to learn the meanings of the count words, one by one. Our account assumes both phylogenetic and ontogenetic continuities and is by far the most parsimonious description of early counting available.

Open questions remain regarding how young infants are able to discriminate between 2 and 3 events, as in Wynn (1996) and Kobayashi et al. (2002). In this case, the Weber fraction and the non-visual nature of the stimuli suggest that neither of the current models accounts for these results. Perhaps object files are less vision- and object-based than we think, and instead are simply a manner for individuating sensory stimuli (be they objects, sounds, events, etc.). Or maybe results of these experiments are somehow an artifact of a developmental shift in the Weber fraction (e.g., see Lipton & Spelke, 2003). The numerical nature of the object file representational system should also be examined more thoroughly. That is, if object files are the dominant representational system employed in small number tasks, it is unclear whether the basis for discrimination is truly numerical or solely object based (and non-numerical). For example, infants may look longer at displays of novel numerosities simply because an open object file no longer has an object to track, and vice versa. Further research should look into these issues. Clearly, data from repeated trials with individual infants are necessary in order to look at individual patterns of variability. In addition, investigations into the developmental trends will also help to shed light on these issues. Through these analyses, the validity of the current models can be assessed and we will gain insight into the nature of preverbal quantity representations and their relation to verbal ones.

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NOTES

1. Gelman and Gellistel (1978, Chapter 11) discuss the relation between the understanding of the cardinal count principle and the principle of one-to-one correspondence.
2. Both Mix et al. (1997) and Moore et al. (1987) claimed to replicate the paradigm used by Starkey et al. (1983) despite severe modifications to the experimental design. Despite these changes, both studies found that infants looked longer at the display with the nonequivalent numerosity–the opposite result of Starkey et al. Despite differing patterns of results, it is clear that all three studies revealed a preverbal attention to the numerosity of the displays and sounds.
3. These results were reported at meetings of the Psychonomics Society in 1999 and the Congress of the International Union of Psychological Science in Sweden in 2000.
4. Response times greater than 2200 ms, shorter than 150 ms, or for incorrect trials were excluded.
5. Barth et al.'s study need not rule out an iterative nonverbal counting process. The response times reported are long (in the neighborhood of 1450 ms). It is possible that these were a function of the decision criterion with the large sets. This may have overshadowed the relatively short time it took for subjects to enumerate (nonverbally count) the two sets. In addition, subjects may have used alternative strategies. For example, these results are consistent with subjects using dot density or overall amount of background area as relevant dimensions, as opposed to number. These alternatives and others need to be explored regarding the iterative or noniterative nature of nonverbal counts.
6. We note that Wynn's (1992a) results can also be accounted for by an object file system of representation.
7. The alternative mapping proposed–a logarithmic one (e.g., Dehaene, 1989)–does allow for unlimited representational capacities. Through a logarithmic mapping, larger values become subjectively closer together, thus preventing representations of arbitrarily large values from imposing arbitrarily large amounts of processing demands. However, when rats or pigeons respond to the difference between two temporal or numerical values (Time Left, Gibbon & Church, 1981; Number Left, Brannon, Wusthoff, Gallistel, & Gibbon, 2001), the data consistently support a linear mapping between objective and subjective quantities.
8. It could be argued that Meck, Church, & Gibbon's (1985) conclusion that the mental magnitude for one count takes about 200 ms contradicts these conclusions, because they indicate that magnitude values are absolute and consistent across individuals (at least within species). Balci & Gallistel (in preparation) show that the 200-ms results may be an artifact of the range of values tested, not an absolute measure of temporal and numerical magnitudes.

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