

Young Natural-Number Arithmeticians

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ABSTRACT—*When preschoolers count to check their arithmetic predictions, their counts are better than when they simply count a set of items on count-only tasks. This is so even for 2 1/2- and 3-year-olds dealing with small values. Such results lend support to the view that learning about verbal counting benefits from a nonverbal count-arithmetic system and challenge theories that place understanding of verbal counting at 4 1/2 or 5 years. That preschoolers readily engage in predicting-and-checking number tasks has implications for educational programs.*

KEYWORDS—*informal cognition; preschool arithmetic; early counting; cognitive development; abstraction; preschool children*

The idea that preschoolers (2 to about 5 years) are budding arithmeticians with respect to the natural (i.e., counting) numbers will strike many as odd. After all, mathematical ideas are not “out there” for inspection. They are abstract ideas, the kind that many assume preschoolers lack. I differ from several theorists who share the view that children younger than 4 or 5 do not understand the cardinal counting principle—the last word in a count represents the cardinal numerosity of the collection—let alone its relationship to addition and subtraction (e.g., Carey, 2004; Fuson, 1988; Mix, Huttenlocher, & Levine, 2002; Piaget, 1952). My viewpoint gains support from arithmetic counting tasks. Young children’s success on such tasks is consistent with the theme that the meaning of the counting procedure is tied to its relation to principles of natural-number arithmetic, as opposed to the idea (expressed by other theorists) that addition and subtraction is an induction that follows after an understanding of counting is achieved. Given the understanding of counting, the children can notice that the repeated placement of one item into a collection increases its value.

For some theorists, children’s understanding of the count numbers requires a conceptual change. For example, Carey

argues that the first few count words (*one, two, three*, etc.) are initially treated like the quantities *one, some, more, a lot, all*, and so on. When children switch to treating count terms above *four* and *five* as unique representations of successively larger cardinal values, they can move on to the idea that each successive count word in the list represents one more than the previous word.

Mix et al. (2002) propose a gradual march from associations of number words to small collections of homogeneous items to associations of number words to larger collections and then to collections of heterogeneous items, and so on. Finally, the child achieves the abstract concept of natural number. Learning about the relation of counting to addition and subtraction follows. Fuson (1988) offers a closely related view. Piaget (1952) requires a qualitative shift from preoperational mental structures to concrete-operational ones, the latter supporting the use of one-to-one correspondences to assess equivalence or nonequivalence of sets, no matter what the set sizes are.

The list of abstraction abilities demonstrated by young children is ever growing, and includes the abilities to assign pairs of look-alike photos, such as a statue and an animal, to their different ontological categories; to reason about unseen causal conditions; to answer questions about the insides of novel items; to attribute mental states to people; and so on (see Gelman & Lucariello, 2002). Still, all this is about people, objects, and events that occur in the natural world, not about arithmetic, which is *really* abstract. Or is it? My inclusion of ideas about the natural numbers and their relation to addition and subtraction was motivated by the ease with which a sample of 5-year-olds who failed a battery of conservation pretests, including those for number, length, and liquid and quantity amounts, responded to training. The conservation tasks assess whether a child understands that a given quantity does not change in amount when it is rearranged or moved, as when one of 2 rows of N items is spread out to look longer than the other, one of 2 sticks is moved to extend beyond the other, one ball of clay is pounded into a flat disk, or one glass of water is poured into a thin, tall beaker (see below for further details). Children younger than 6 years of age typically failed to conserve on any of these tasks. Yet they had no trouble choosing two of three rows of objects that had the same or

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different number. Similarly, they did very well on length problems for which they had to choose two of three sticks that were either the same or different lengths as the experimenter continually altered their configuration (Gelman, 1969). The children also achieved high levels of success on posttests of conservation of number and length, as well as of liquid and mass amount. It seemed most unlikely that training instilled a new set of mental structures that would support belief in equivalence despite the contradiction of perceptual data. More plausibly, the children were ready to conserve number across irrelevant transformations as well as conserve the pretested other quantities.

My search was on for ways to show early number knowledge and other abstract abilities. As supporting data came in, I started to move toward a domain-specific view of cognitive development, with the assumption that the presence of a small set of implicit mental structures support learning on the fly. The idea that the mind initially has a small set of skeletal structures, each with their own principles, contrasts with the initial-blank-slate view held by associationists and Piaget. Despite their skeletal format, they are mental structures, and we know from cognitive psychology that when one already has a domain-specific mental structure, it is rather easy to learn more about content that fits that structure. The mind will use these different structures to actively engage and assimilate environmental data that share their internal relations.

How does this bear on the acquisition of verbal knowledge of counting and arithmetic? It is known that a nonverbal numerical-reasoning system that supports counting, addition, and subtraction is available to young children, infants, and animals (Barth, La Mont, Lipton, & Spelke, 2005; Cordes & Gelman, 2005). This supports Gelman and Gallistel's (1986) proposal that implicit principles of nonverbal enumeration facilitate identifying and learning about both the words for counting and rules for how to use them. The nonverbal principles—one and only one unique tag for each to-be-counted item, stable ordering of the tags, and

use of the last tag to represent the cardinal value—correspond to the use rules of verbal counting. So, given a counting list that maps to nonverbal counting principles, the stage is set for the child's nonverbal structure to identify and start to assimilate the count list to a relevant structure. Since verbal counts and numerical estimates do not stand alone—that is, they gain their meaning from their relation to a mental structure of arithmetic reasoning that includes principles of addition, subtraction, and ordering—assessment of understanding of the language of counting might best be tapped in arithmetic counting tasks as opposed to tasks that only assess counting ability.

SOME EARLY RESEARCH

Piaget (1952) designed tasks that reflected his goal of contributing results to the efforts of logicians like Russell and Von Neumann to show that arithmetic could be derived from set theory. The number-conservation task taps the idea that any two sets that can be placed in one-to-one correspondence are equal—whatever their arrangement or set size. In the conservation task, even preschool children agree that when the items in two rows are opposite each other, the numbers are the same. But, when the items in one row are spread out, preschoolers deny the conservation of equivalence, typically saying the spread-out row has more. Failures on the task contributed to Piaget's conclusion that preschoolers lack a concept of natural number. For him, early counting is just rote learning.

I found it hard to entertain the idea that the conservation-training experience I offered could generate a new set of mental structures. The children must have known quite a bit already. A series of magic tasks (detailed in Fig. 1 and summarized in Gelman & Gallistel, 1986) provided the first lines of evidence. In phase 1, a child saw two plates with varying numbers of items on them, such as toy mice or strawberries. One plate was dubbed “the winner” and the other “the loser.” The set-up (phase 2),



Fig. 1. The three phases of the number-magic tasks. In phase 1 (left), two plates with toy red strawberries on them were labeled the “winner” and the “loser,” and children were rewarded with prizes for identifying the “winner.” In phase 2 (center), the plates were covered and the covers were shuffled; children then pointed to the cover that they thought hid the winner-plate; if they were incorrect on the first try, they were allowed to pick the second cover; they were always rewarded whenever they identified the winner. At various points during phase 2, children were asked why a given plate was a winner or loser. At first, they seldom said anything; but as we continued, most of the 3-, 4-, and 5-year-olds in the study talked about number correctly during this phase. Children also became sufficiently interested in their collection of prizes to stop watching closely as the experimenter shuffled the covers, allowing the displays to be surreptitiously altered in number of strawberries or in non-numerical values like length, color, and so on. Across experiments, changes in the number or arrangement of items in phase 3 (right) were noticed by almost all children in some way. Additionally, in almost all children, including the 2 1/2-year-olds, changes in number elicited strong emotional reactions, search for missing items, and verbal statements about the number change. (The right panel shows a child with both sets of fingers in her mouth; before this, she was smiling, having found the “winner.”)

which was modeled after a shell game, followed as a series of trials in which the plates were covered and shuffled as the child watched. Then the child made guesses about which cover concealed the “winner” plate. The experimenter avoided talk about number or quantity. Phase 3 began after both of the plates were uncovered by the child, who now saw the effect of a surreptitious addition or subtraction of one or more items or a change in how close together the items were on a plate. Changes in number in the subtraction conditions were greeted with surprise, search, and talk about the missing items. For example, J.E., who expected three mice on one plate and two on the other but saw two on both plates during phase 3, told us that we “took one. Cuz, there’s two now . . . where is it?” J.S., who expected two and encountered three, explained the unexpected addition as follows: “Something might pop out of these little bumps” (on the inside of the cans used to cover the plates) as a possible source of the additional mouse. Changes in length or density were accepted—for example, “because it’s still three.”

As the hiding-and-guessing trials in phase 2 proceeded, children began to volunteer that number mattered—for example: “That’s the winner, three—one, two, three,” or “one, two—loser.” Of particular interest was the use of rule-governed but idiosyncratic count lists, as when D.S., who was 2 1/2 years old, encountered three mice instead of the expected two; he volunteered “umm, one, two” for the two-mouse plate and “one, two, six” for the three-mouse plate and then the same “one, two, six” count at a later point in phase 3. Analyses in Gelman and Gallistel (1986) of errors across counting and magic experiments revealed that children who used idiosyncratic count lists (e.g., “one, two, six” instead of “one, two, three”) across trials made many fewer counting errors than those who used the conventional count sequence.

SOME RECENT RESEARCH

Bisanz, Sherman, Rasmussen, and Ho (2005) confirmed and extended our conclusion that young children correctly negotiate arithmetic problems when the total number is equal to or less than four. But many, including Carey (2004), counter that judgments of numerical order and equality for very young children are mediated by mechanisms that work only for this range (Carey, 2004). When we tried to use our magic paradigm with larger sets, we had to use larger cans to cover the displays with larger numbers of items. They were so large that children dropped them or put them on their heads or faces to play games like “hats,” “masks,” etc. A different task was needed. Zur and Gelman (2004) hit upon one that was inspired by a familiar ditty: *There were six in the bed and the little one said “roll over,” and they all rolled over, one fell out, and there were five in the bed, and the little one said “roll over”*—and so on. To start, we worked mainly with 4-year-olds. Then a suitable variant of the arithmetized counting task was run with 3-year-olds.

To begin one set of trials, the experimenter handed the child objects—say, nine “donuts” made from felt—to place in a “bakery” and asked: “How many donuts are in the bakery?” The child answered, and the adult continued: “I have 3 three pennies to buy donuts. Predict, how many will be left.” Charmingly, children assumed that one penny buys one donut. The purchase complete, the experimenter continued: “Check how many you have now.” Children now counted, and almost always they were correct. When rare counting errors occurred, a recount was almost always correct, even for the largest values. The achieved checking value served as the first value for the next cycle of prediction and checking: “Okay, so now you have six donuts. Here comes a delivery of two donuts. Predict how many will be in the donut shop,” and so on.

Using videotapes, we verified that children almost always predicted without counting but counted aloud when checking. They did not try to make their counts conform to their predictions and acted as if verbal counting was definitive. We assumed that their prediction would benefit from their nonverbal number knowledge and the mapping from this to number words. This is consistent with the proposal that preschoolers can perform both approximate and precise counting-arithmetic tasks (Barth et al., 2005; Cordes & Gelman, 2005).

The task was developed in the context of a classroom. When it was clear that the children in the class succeeded as a group, even on totals of 15, we tested each of them individually. A year later we tested a novice group of same-aged children. As expected, there was some effect of experience. Still, even the children who came naive to the task did well. Figure 2 plots the Zur and Gelman (2004) data for one problem set, showing the averages and ranges of the children’s predictions as a function of the correct answers. Predictions increased systematically even

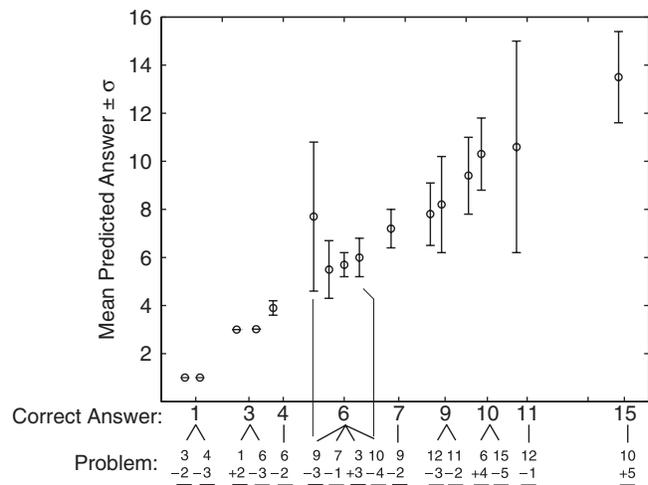


Fig. 2. The relationship between the correct answers for arithmetic problems and the prediction averages and ranges (denoted by bars) given by 14 children with ages ranging from 3.11 to 5.4 years (average age = 4.7). Note the tendency to give the same prediction for different problems having the same arithmetic result (e.g., the problems with an answer of 6), a clue that implicit rules of arithmetic were available.

when the correct answers were high—between 10 and 15! Further, predictions for problems with the same answer tended to cluster together (e.g., $3 + 3$, $7 - 1$, and $10 - 4$). This is consistent with our assumption that predictions are mediated by the mapping of a nonverbal arithmetic system to a verbal one. The 3-year-olds did rather well at predicting and checking when answers were as large as 5. Eighty percent of the predictions were in the right direction; 92% of the checking trials were correct. Thus, we have more evidence for the proposal that the arithmetized counting system is applied to small numbers as well as to larger ones.

An unexpected result led us to propose that preschool children implicitly recognize the inverse principle (if $a + b = c$, then $c - b = a$, and vice versa): There was a reliable tendency for them to do better on the second of a pair of inverse problems—as they did on the problem $10 - 6$ when they had already encountered the problem $6 + 4$, as opposed to when they had seen $5 + 4$. This corroborates and extends Bryant, Christie, and Rendu's (1999) demonstration of 5- to 6-year-olds' success on the inverse principle with problems of the form $a + b - b$.

There is other evidence that young children have *some* implicit knowledge of arithmetic before they master the counting list of their language. Recall that 2 1/2-year-olds in a magic task understood the effect of addition and subtraction on numerical ordering (Barth et al., 2005, report related findings). A similar conclusion follows from Sarnecka and S. Gelman's (2005) clever use of addition and subtraction questions in their study of 4-year-olds' knowledge of counting and quantifier terms. These researchers added to (or subtracted from) a can they first said had either a specific number (e.g., "six") or a nonspecific quantity (e.g., "a lot"). When asked whether the amount following an addition or subtraction was different, the children said "yes" for the number-word can and "no" for the quantifier one. When a can was shaken, the same children said that neither value had changed. This effect of framing questions in an arithmetic format provides more support for the position that young children know that the meaning of count words is related to addition and subtraction. The results also fit with evidence that early language learners take number words to refer only to cardinal values and not to imprecise quantities that can be referred to with quantifiers like "some" and "all" (Hodent, Bryant, & Houdé 2005; Hurewitz, Papafragou, Gleitman, & Gelman, 2006). The Hodent et al. (2005) study follows up the finding that when French-speaking children are provided arithmetic stories that represent $1 + 1 = 1$ and $1 + 1 = 3$, they say the former is "not okay" and the latter is "okay." Hodent et al. reasoned that this could reflect the ambiguity of *un* in French, where it represents both the first count word (i.e., "one") and the indefinite singular in the pair *un-des* ("a"—"some"). To control for this possibility, there were conditions in which 2-year-old French speakers were shown two items that were then covered. Next, one item was added to the hidden display. Subsequently, uncovered test trials included displays of two, three, or four items. Now, the children declared

that the outcome of three was "okay" but that the displays with two or four items were "not okay." Hurewitz, Papafragou, Gleitman, and Gelman (2006) show that that, when presented with displays that can be interpreted as either "two" or "four" or as either "some" or "all," very young children do better at labeling the displays when requested to use a number than they do when asked to label the displays with quantifiers. Sarnecka and S. Gelman's (2005) study with older preschoolers continues this pattern of evidence, favoring the position that children know that the operations of adding and subtracting systematically increase and decrease the value of cardinality, even if they cannot reliably count the number of items involved. So too does Hartnett and Gelman's (1998) study on 5- to 8-year-olds' understanding of the successor principle. These children were more likely to say that adding 1—as opposed to counting further—could go on indefinitely. Why? Although they knew that adding would increase the number, they did not know the words for those numbers, as illustrated by explanations two children gave: "You just can't put it in the newspaper like, we thought of some new numbers"; and "There aren't real numbers [meaning known terms] but you could make them up."

It would be a mistake to assume that the eventual command of the verbal counting system guarantees ready understanding of all number concepts and other mathematical ideas. Indeed, many adults lack an understanding of the numbers used in the discourse of commerce, like 12.56 million or 2 trillion dollars, or that the birthrate of a given country is 1.9 and not the whole number 2, probably because they have no reason to have developed a quantitative representation of such large sums of money or rational numbers (fractions). Learning about fractions in either fractional or decimal format turns out to be extremely difficult, even for many adults. In fact, the relative ease of working with the counting numbers turns out to be a key source of difficulty regarding rational numbers. This is because the principles underlying each kind of number differ dramatically. Each natural number has a unique next. By contrast, there is an infinite number of rational numbers between each pair of rationals. The fact that a unit fraction's value increases as the value of its denominator decreases is a major source of trouble for students who believe numbers are what one gets when one counts (Hartnett & Gelman, 1998).

CONCLUSION AND FURTHER DIRECTIONS

When children as young as 2 1/2 years participate in tasks that evoke arithmetic reasoning, they reveal knowledge that counting determines cardinality. Still, their failure on stand-alone counting tasks must be explained, for both pedagogical and theoretical reasons. Acquisition of the specific language of the cardinal count words and the particular notation system for arithmetic needs to be researched. On the practical side, when we take the predicting-and-checking tasks used in our research to the classroom, children enjoy and readily engage in the

activities, including when they predict and check the number of seeds they will find inside different fruits. Further, on the assumption that part of the difficulty of understanding fractions is due to the lack of extensive experience with measurement tools and their corresponding variations in unit size, we have moved to offering relevant experiences in our preschool science program.

Recommended Reading

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