First-order Multi-Modal Deduction[1]

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Summary

We study prefixed tableaux for first-order multi-modal logic, providing proofs for soundness and completeness theorems, a Herbrand theorem on deductions describing the use of Herbrand or Skolem terms in place of parameters in proofs, and a lifting theorem describing the use of variables and constraints to describe instantiation. The general development applies uniformly across a range of regimes for defining modal operators and relating them to one another; we also consider certain simplifications that are possible with restricted modal theories and fragments.

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1 Introduction

Recent years have seen an explosion in research in formalizing inference in modal logic [Goré, 1999, Basin et al., 1998, Fitting and Mendelsohn, 1998] and in using modal theories in knowledge representation [Fagin et al., 1995, McCarthy and Buvač, 1994, Stone, 1998]. Unfortunately, research on modal inference does not link up as directly as could be hoped with proposed modal theories. This report aims to help provide such links by providing a set of extremely general results about first-order multi-modal deduction in terms of analytic tableaux and a prefix representation of possible worlds. We first provide sound and complete ground tableau and sequent inference systems, extending and refining those presented in [Fitting and Mendelsohn, 1998] to the multi-modal case. Then we show how to apply general proof-theoretic techniques to derive an equivalent calculus where Herbrand terms streamline proof search [Lincoln and Shankar, 1994]. Finally, we derive a *lifted* multi-modal sequent inference system, which uses unification (or constraint-satisfaction) to resolve the values of variables, in the style of [Voronkov, 1996]. From one point of view, this report can be regarded as the multi-modal generalization of the results presented for linear logic and first-order modal logic in [Lincoln and Shankar, 1994, Fitting, 1996, Fitting and Mendelsohn, 1998]; alternatively, it can be seen as recasting into a modal setting the results of [Stone, 1999b], which investigates first-order intuitionistic logic along similar lines.

Formal modal logic goes back eighty years [Lewis, 1918, Lewis and Langford, 1932]. Yet according to McCarthy [McCarthy, 1997], for example, the modal logic literature still does not offer a formalism with the intensional expressive power—including fresh modalities defined *ad hoc*, and means to describe *knowing what* by concise and easily manipulated formulas—that is needed for knowledge representation in Artificial Intelligence. Moreover, typical results from the modal logic literature do not support the design of specialized modal inference mechanisms to solve particular knowledge representation tasks.

The approach adopted here is a response to these gaps. We tackle a first-order multi-modal logic with an arbitrary number of modal operators and a flexible regime for relating different modal operators to one another—this gets at limitations in expressive power. We consider inference in analytic tableaux (or, seen upside-down, in the cut-free sequent calculus)—this provides a close grounding with techniques for implementing deduction. And—in order to suggest and facilitate results about specialized inference algorithms, such as [Stone, 1999a, Stone, 1999c]—we avoid definitions for logical connectives, we represent worlds using *prefix* terms, denoting *paths of accessibility* among possible worlds, and we factor out reasoning about accessibility into side conditions on inference rules.

Individually, these choices are familiar from research on modal logic. For example, [Fitting and Mendelsohn, 1998] present a comprehensive treatment of the first-order modal logic using prefix terms and analytic tableaux. But they treat only a single modal operator. [Basin et al., 1998] adopt a proof-theoretic view of first-order modal logic in which reasoning about accessibility and boolean reasoning are clearly distinguished. But they also treat only a single modal operator, and do so using atomic terms for worlds and natural deduction proof. Meanwhile, [Baldoni et al., 1998] explore a first-order multi-modal logic with related operators using analytic tableaux—but they also use atoms to refer possible worlds and now allow reasoning about accessibility to interact with first-order reasoning. [Nonnengart, 1993] covers a few multi-modal logics using a prefix representation of worlds, but avoids interactions among modal operators and advocates a translation method with purely classical reasoning where modal proofs are difficult to study.

Thus the combination of ideas explored here—a combination that plays a crucial role for applications in logic programming and knowledge representation—remains a novel one. In fact, even today, research in modal logic—whether the investigation is more mathematical [Goré, 1992, Massacci, 1998b, Massacci, 1998a, Goré, 1999] or primarily concerns algorithms for proof search [Otten and Kreitz, 1996, Beckert and Goré, 1997, Schmidt, 1998]—is dominated by the study of the propositional logic of a single modal operator (or accessibility relation).

When multiple modal operators are considered, their interpretations and interactions are often predefined. In PDL and terminological logics we have combinations of orthogonal K modal operators [Goldblatt, 1992, Schild, 1991]. In typical epistemic logics, we have orthogonal combinations of S5 modal operators to model knowledge [Fagin et al., 1995] or KD45 to model belief [Nonnengart, 1993]. In tense logics, we have a predefined pair of symmetric operators for past and future [Prior, 1967].

Other researchers who have investigated modal logic in a first-order setting have tended to jump directly into a discussion of particular theorem-proving strategies, particularly resolution [Jackson and Reichgelt, 1987, Wallen, 1990, Catach, 1991, Frisch and Scherl, 1991, Auffray and Enjalbert, 1992, Ohlbach, 1993]. Often the modal component of the language is translated away using first-order quantifiers as soon as a semantics for it is provided. Another strategy has been to study only logic programming proof for multi-modal logic [Fariñas del Cerro, 1986, Baldoni et al., 1993, Baldoni et al., 1996]. For such approaches, it suffices to provide soundness and completeness proofs for a restricted logical fragment; indeed, these approaches often avail themselves of specialized proof-theoretic representations that do not generalize to the full modal language and whose relationship to the general proof-theory is left unexplored.

2 Ground First-order Multi-Modal Deduction

These preliminaries have suggested that there is both the motivation and the need to study prefixed tableaux for multi-modal logics in a general way. So let us consider the syntax, semantics and ground proof theory of a broad multi-modal language.

2.1 Syntax

Our language is defined by a *signature* $\langle$ OP, REL, VAR, CONST $\rangle$. OP is a finite collection of *modalities*, which we shall write using the *necessity* operators $\Box_1, \ldots, \Box_k$ and the *possibility* operators $\Diamond_1, \ldots, \Diamond_k$. These modal operators may be subject to any of a number of logical conditions, which we explain presently. REL is a countable set of relation symbols $R_1, \ldots$ each of which is specified for arity; REL should contain at least one relation but may be finite. VAR is a countably infinite collection of variables $x_1, \ldots$, and CONST is a countable collection of constant symbols $c_1, \ldots$ possibly finite but containing at least one symbol. Thus, we insist on a countable language here. (Both constants and variables will be interpreted rigidly.) While OP, REL and VAR can remain fixed throughout the rest of this report, it will be convenient to consider languages in which a countably infinite number of *parameters* are included in the language to supplement the symbols in CONST. So we write $L(C)$ to describe the language built over the signature $\langle$ OP, REL, VAR, C $\rangle$. The basic language is then $L(\text{CONST})$.

Definition 1 (Formulas) *The set of formulas in $L(C)$ is the smallest set meeting all of the following criteria. If R_i is a l-ary relation symbol of REL and $t_1, \ldots, t_l$ is a sequence of length l each of whose elements is some constant c_j in C or some variable x_j in VAR, then $R_i(t_1, \ldots, t_l)$ is a formula. $\top$ is also a formula. If A and B are formulas and i indexes one of the k modalities of OP, then the formulas also include $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$, $\Box_i A$ and $\Diamond_i A$. If A is a formula and x is a variable of VAR, then $\forall xA$ and $\exists xA$ are also formulas.*

We appeal to the usual notions of *free* and *bound* occurrences of variables in formulas; we likewise invoke the *depth* of a formula (the largest number of nested logical connectives in the formula).

Definition 1 is set up so that different formulas are independent, which is convenient if we intend to apply our results to restricted logical fragments. An alternative approach invites us to take $A \vee B$ as an abbreviation for $\neg(\neg A \wedge \neg B)$, to take $A \supset B$ as an abbreviation for $\neg(A \wedge \neg B)$, to take $\Diamond_i A$ as an abbreviation of $\neg \Box_i \neg A$ and to take $\exists xA$ as an abbreviation for $\neg \forall x \neg A$. Such abbreviation cuts down the cases we must consider in the metatheory—an alternative streamlining device is to introduce *uniform notation* as in [Smullyan, 1968, Smullyan, 1973, Fitting, 1983, Wallen, 1990] to collapse proof rules without collapsing connectives. From a pedagogical point of view, not much hinges on this; when analysis of numerous similar cases is required, proofs are rarely presented with explicit analysis for all of them.

2.2 Semantics

As is standard, we describe the models for the modal language in two steps. The first step is to set up *frames* that describe the structure of any model; a full model can then be obtained by combining a frame with a way of assigning interpretations to formulas in a frame.

Definition 2 (frame) *A first-order k-*frame *(or, here, simply frame) consists of a tuple* $\langle G, R, D\rangle$ *where:* G *is a non-empty set, whose members are generally called* possible worlds*;* R *names a family of binary relations on* G*,* R_i *for* $i \leq k$*, generally called* accessibility relations*; and* D *is a function, called the* domain function *mapping members of* G *to non-empty sets.*

Within the frame F, the function D induces a set $D(F)$, called the *domain of the frame*, as $\cup\{D(w) \mid w \in G\}$. In order to simplify the treatment of constant symbols, it is also convenient to define a set of objects that all the domains of the different possible worlds have in common, the *common domain of the frame* F: $C(F) = \cap\{D(w) \mid w \in G\}$. We effectively insist that $C(F)$ be non-empty as well, since CONST is non-empty and each symbol in CONST must be interpreted by an element of $C(F)$.

The intermediate level of frames is useful in characterizing the meanings of modal operators and modal quantification. In particular, simply by putting constraints on R_i or on D at the level of frames, we can obtain representative classes of models in which certain general patterns of inference are validated. The constraints we will avail ourselves of are introduced in Definition 3.

Definition 3 *Let* $\langle G, R, D\rangle$ *be a first-order k-*frame*. We say the frame is:*

- reflexive *at i if* wR_iw' *for every* $w \in G$*;*
- symmetric *at i if* wR_iw' *only if* $w'R_iw$ *for every* $w, w' \in G$*;*
- transitive *at i if, for any* $w, w'' \in G$*,* wR_iw'' *whenever there is a* $w' \in G$ *such that* wR_iw' *and* $w'R_iw''$*;*
- serial *at i if for each* $w \in G$ *there is some* $w' \in G$ *such that* wR_iw'*;*
- euclidean *at i if whenever* wR_iw' *and* wR_iw'' *then* $w'R_iw''$ *for any* $w, w', w'' \in G$*;*
- narrowing *from j to i if* wR_iw' *implies* wR_jw' *for all* $w, w' \in G$*;*
- constant domain *if* $D(w) = D(w')$ *for any* $w, w' \in G$*;*
- increasing domain *if for all* $w, w' \in G$*,* $D(w) \subseteq D(w')$ *whenever there is some accessibility relationship* wR_iw'*.*

Our scheme for using the constraints of Definition 3 depends on establishing a regime for the k modal operators in the language, describing the inferences that should relate the modal operators. The regime is defined as follows.

Definition 4 (Regime) *A first-order k-*regime *(or, here, simply regime) is a tuple* $\langle A, N, Q\rangle$*, where:* A *is a function mapping each integer in the interval* $[1..k]$ *into one of the symbols K, KB, K4, K5, K45, KD, KDB, KD4, KD5, KD45, T, B, S4 and S5;* N *is a (strict) partial order on the integers in the interval* $[1..k]$*, and* Q *is one of* constant, increasing, *or* varying.

The reader will recognize the symbols in the image of A as the classic names for modal logics of a single modality. This meaning for these symbols can be enforced by considering only frames that *respect* the given *regime*.

Definition 5 (Respect) *Let* $F = \langle G, R, D\rangle$ *be a first-order k-frame, and let* $S = \langle A, N, Q\rangle$ *be a first-order k-regime. We say* F respects S *whenever the following conditions are met (taking i and j to range over all integers in the interval* $[1..k]$*):*

- *If* $A(i)$ *is T, B, S4, or S5 then* R_i *is reflexive.*
- *If* $A(i)$ *is KB, KDB, or B then* R_i *is symmetric.*
- *If* $A(i)$ *is K4, K45, KD4, KD45, S4 or S5 then* R_i *is transitive.*
- *If* $A(i)$ *is KD, KDB, KD4, KD5 or KD45 then* R_i *is serial.*
- *If* $A(i)$ *is K5, K45, KD5, KD45 or S5 then* R_i *is euclidean.*
- *If* $i \leq j$ *according to* N *then* F *is narrowing from j to i.*
- *If* Q *is* constant, *then* F *is constant domain; if* Q *is* increasing, *then* F *is increasing domain.*

From now on, we assume that some regime $S = \langle A, N, Q\rangle$ is fixed, and restrict our attention to frames that respect S. Informally, now, a model consists of a frame together with an interpretation.

Definition 6 (interpretation) J *is an* interpretation *in a first-order k-frame* $\langle G, R, D\rangle$ *if J satisfies these two conditions:*

1. J *assigns to each n-place relation symbol* R_i *and each possible world* $w \in G$ *some n-place relation on the domain of the frame* $D(F)$.
2. J *assigns to each constant symbol c some element of the common domain of the frame* $C(F)$.

Thus we can define a *model* over S thus:

Definition 7 (model) *A first-order k-*modal model *over a regime* S *is a tuple* $\langle G, R, D, J\rangle$ *where* $\langle G, R, D\rangle$ *is a first-order k-frame that respects S and J is an interpretation in* $\langle G, R, D\rangle$.

To define truth in a model, we need the usual notion of assignments and variants:

Definition 8 (assignment) *Let $\mathcal{M} = \langle G, \mathcal{R}, D, J\rangle$ be a first-order k-modal model (over some regime S). An* assignment *in $\mathcal{M}$ is a mapping g that assigns to each variable x some member $g(x)$ of the domain of the frame of the model $D(\langle G, \mathcal{R}, D\rangle)$.*

On occasion, we will be asked to interpret formulas not just in the ordinary language $L(C)$ with a given set of model operators, relations, constants and variables, but in an expanded language $L(C \cup P)$ which also includes a set P of first-order *parameters*; we will want to use the same models for this interpretation. Over $L(C \cup P)$, we suppose that an assignment in $\mathcal{M}$ also assigns some member $g(p)$ of the domain of the frame of $\mathcal{M}$ to each parameter p in P. This allows the model $\mathcal{M}$ to provide an interpretation of formulas in $L(C \cup P)$, without the interpretation J of $\mathcal{M}$ describing P.

Definition 9 (variants) *Let g and g′ be two assignments in a model $\mathcal{M} = \langle G, \mathcal{R}, D, J\rangle$; g′ is an* x-variant of g at a world $w \in G$ *if g and g′ agree on all variables except possibly for x and $g'(x) \in D(w)$.*

Definition 10 (truth in a model) *Let $\mathcal{M} = \langle G, \mathcal{R}, D, J\rangle$ be a first-order k-modal model. Then the formula A is* true at world *w* of model $\mathcal{M}$ on assignment *g—written $\mathcal{M}, w \Vdash_g A$—just in case the clause below selected by syntactic structure of A is satisfied:*

- *A is $\top$: Then always $\mathcal{M}, w \Vdash_g A$.*
- *A is $R_i(t_1, \ldots, t_n)$: Then $\mathcal{M}, w \Vdash_g A$ just in case $\langle e_1, \ldots e_n\rangle \in J(R_i, w)$, where for each t_i, e_i is $J(t_i)$ if t_i is a constant and $g(t_i)$ if t_i is a variable (or a parameter).*
- *A is $\neg B$: Then $\mathcal{M}, w \Vdash_g A$ just in case $\mathcal{M}, w \nVdash_g B$.*
- *A is $B_1 \wedge B_2$: Then $\mathcal{M}, w \Vdash_g A$ just in case both $\mathcal{M}, w \Vdash_g B_1$ and $\mathcal{M}, w \Vdash_g B_2$.*
- *A is $B_1 \vee B_2$: Then $\mathcal{M}, w \Vdash_g A$ just in case either $\mathcal{M}, w \Vdash_g B_1$ or $\mathcal{M}, w \Vdash_g B_1$.*
- *A is $\Box_i B$: Then $\mathcal{M}, w \Vdash_g A$ just in case for every $w' \in G$, if $w\mathcal{R}_i w'$ then $\mathcal{M}, w' \Vdash_g B$.*
- *A is $\Diamond_i B$: Then $\mathcal{M}, w \Vdash_g A$ just in case there is some $w' \in G$ such that $w\mathcal{R}_i w'$ and $\mathcal{M}, w' \Vdash_g B$.*
- *A is $\forall x B$: Then $\mathcal{M}, w \Vdash_g A$ just in case for every x-variant g′ of g at w, $\mathcal{M}, w \Vdash_{g'} B$.*
- *A is $\exists x B$: Then $\mathcal{M}, w \Vdash_g A$ just in case there is some x-variant g′ of g at w such that $\mathcal{M}, w \Vdash_{g'} B$.*

Where the semantic definition considers variant assignment, the proof systems must appeal to syntactic substitution. Thus, for soundness, we need to establish an appropriate relationship between assignments and substitutions.

Lemma 1 (Substitution) *Suppose $\mathcal{M} = \langle G, \mathcal{R}, D, J\rangle$ is a first-order k-modal model for the language $L(C)$, $w \in G$, and g_1 and g_2 are two assignments in $\mathcal{M}$ for the language $L(C \cup P)$. Suppose A is a formula in $L(C \cup P)$ in which the symbol x may have some occurrences and the symbol y either does not occur or is a constant. (We are neutral as to whether x is drawn from VAR or P, and whether y is drawn from VAR, P or CONST.) Write $A[y/x]$ for the result of replacing all (free) occurrences of x with occurrences of y. Finally, suppose g_1 and g_2 agree on all the parameters and free variables of A except possibly for x, and either $g_1(x) = g_2(y)$ or $g_1(x) = J(y)$ (according to the category of y). Then*

$$\mathcal{M}, w \Vdash_{g_1} A \Leftrightarrow \mathcal{M}, w \Vdash_{g_2} A[y/x]$$

Proof. By induction on the structure of formulas. The base case has A atomic; $\top$ is obvious; so we just need to show $\mathcal{M}, w \Vdash_{g_1} R_i(t_1, \ldots, t_n) \Leftrightarrow \mathcal{M}, w \Vdash_{g_2} (R_i(t_1, \ldots, t_n))[y/x]$. This follows because the arguments of R_i induce the same tuple $\langle e_1, \ldots, e_n\rangle$ in both cases. Each t_j is either a constant, x, or a variable or parameter other than x. For constants, $t_j = t_j[y/x]$ and $e_i = J(t_j) = J(t_j[y/x])$. For x, $x[y/x] = y$ and (as appropriate) $e_i = g_1(x) = g_2(y)$ or $e_i = g_1(x) = J(y)$. For variables or parameters other than x, $t_j[y/x] = t_j$ and $e_i = g_1(t_j) = g_2(t_j)$.

Now assume the lemma true for formulas of depth N or less, and consider a formula A of depth $N+1$. We illustrate the argument for representative cases depending on whether A is composed by a boolean operation, a modal operator, or a quantifier.

Booleans. Suppose A is $B_1 \wedge B_2$. The induction hypothesis gives $\mathcal{M}, w \Vdash_{g_1} B_1 \Leftrightarrow \mathcal{M}, w \Vdash_{g_2} B_1[y/x]$ and $\mathcal{M}, w \Vdash_{g_1} B_2 \Leftrightarrow \mathcal{M}, w \Vdash_{g_2} B_2[y/x]$. It then follows from the truth-definition for $B_1 \wedge B_2$ that $\mathcal{M}, w \Vdash_{g_1} A \Leftrightarrow \mathcal{M}, w \Vdash_{g_2} A[y/x]$.

Modals. Suppose A is $\Diamond_i B$. Either $\mathcal{M}, w \Vdash_{g_1} A$ or not. If so, then there is a $w' \in G$ with $w \mathcal{R}_i w'$ and $\mathcal{M}, w' \Vdash_{g_1} B$. By induction hypothesis $\mathcal{M}, w' \Vdash_{g_2} B[y/x]$ and hence $\mathcal{M}, w \Vdash_{g_2} A[y/x]$. Otherwise there is no $w' \in G$ with $w \mathcal{R}_i w'$ and $\mathcal{M}, w' \Vdash_{g_1} B$. By induction hypothesis, then, there cannot be any $w' \in G$ with $w \mathcal{R}_i w'$ and $\mathcal{M}, w' \Vdash_{g_2} B[y/x]$. So it is also not the case that $\mathcal{M}, w \Vdash_{g_2} A[y/x]$.

Quantifiers. Suppose A is $\exists v B$. We define B^* so that $A[y/x]$ is $\exists v B^*$. (There are two cases: $x = v$ and $x \neq v$. In the first case, $A[y/x] = \exists v B$ and $B^* = B$; in the second, $A[y/x] = \exists v(B[y/x])$ and $B^* = B[y/x]$.) Now, $\mathcal{M}, w \Vdash_{g_1} A$ is equivalent to the condition that there is a v-variant g_1' of g_1 with $\mathcal{M}, w \Vdash_{g_1'} B$. I claim that exactly when there is such a g_1', there is a v-variant of g_2' of g_2 with $g_2'(v) = g_1'(v)$ and $\mathcal{M}, w \Vdash_{g_2'} B^*$. From this claim, the lemma follows. To show the claim for $B^* = B$, observe that g_1' and g_2' agree on all the parameters and free variables of B^*; the induction hypothesis applies (for any substitution of elements neither of which occurs in B) to establish the claim. Alternatively, for $B^* = B[y/x]$, the free variables of B are those of A plus

v, and by construction g'_1 and g'_2 agree on v. So the induction hypothesis applies for the substitution of y for x to establish the claim. ■

By a *sentence* we mean a formula of $L(\text{CONST})$ in which no variables occur free. For any sentence A, model $\mathcal{M}$ and world w of $\mathcal{M}$, Lemma 1 guarantees that $\mathcal{M}, w \Vdash_g A$ for some assignment g in $\mathcal{M}$ exactly when $\mathcal{M}, w \Vdash_g A$ for all assignments g in $\mathcal{M}$. In this case, we can write simply $\mathcal{M}, w \Vdash A$ and say that A is *true in* $\mathcal{M}$ *at* w.

Definition 11 (Valid) *Let A be a sentence and $\mathcal{M} = \langle G, \mathcal{R}, D, J \rangle$ be a first-order k-modal model. A is* valid in $\mathcal{M}$ *if for every world $w \in G$, $\mathcal{M}, w \Vdash A$. A is* valid *(on the regime $\langle \mathcal{A}, \mathcal{N}, \mathcal{Q} \rangle$) if A is valid in any model $\mathcal{M}$ that respects the regime.*

2.3 Proof theory

The most concise and general systems for modal proof are *prefixed tableaux*, which originate in the work of Fitting [Fitting, 1972, Fitting, 1983]. The idea is to associate formulas in proofs with concise terms, *prefixes*, which identify a world in a model. Formally, we assume a countable set κ of *modal parameters*, enumerated $\alpha_1, \alpha_2, \ldots$. (When the enumeration of κ is not important, I will also write its elements α, β, etc.)

Definition 12 (Prefix) *A* prefix *is a finite sequence of modal parameters. I will use ε to denote the empty prefix and μ, ν, etc., to denote general prefixes. A* prefixed formula *is an expression of the form A^μ where μ is a prefix and A is a formula.*

There are departures from Fitting's notation here, but not the essential ideas. We use modal parameters rather than integers to establish an exact parallel with reasoning with first-order parameters and substitutions; to fit with proof theory more generally, we reserve the symbol σ for substitutions and reserve the 'prefix' position l on formulas—$l\,A$ or $l : A$—for the assocation between formulas in deductions and proof-terms indicating how those formulas contribute to the deduction (as used for example in establishing correspondences between tableau or sequent proofs and natural deduction proofs; see for example [Gallier, 1993]).

Our proof rules will work with *signed prefixed formulas*.

Definition 13 (Signed expressions) *If E denotes the expressions of some class, then the* signed *expressions of that class are expressions of the form* **t***e or* **f***e for e an expression drawn from E. We use* **u** *as a metavariable over* **t** *and* **f**.

The use of prefixes gives us the need to talk about the language $\Pi(\kappa)$ of prefixes over the signature κ of modal parameters, and the language $L(C)^{\Pi(\kappa)}$ of prefixed formulas with formulas drawn from $L(C)$ and prefixes drawn from $\Pi(\kappa)$. In fact, as mentioned earlier, the proof rules will also assume a set P of first-order parameters, so that proofs will contain signed expressions drawn from $L(\text{CONST} \cup P)^{\Pi(\kappa)}$.

In tandem with signed prefixed formulas, we will also need *typings* that specify which accessibility relations different transitions between prefixes instantiate, and which possible worlds different individuals exist at.

Definition 14 (Typing) *A* typing *(over a language L^Π) is a set Σ of statements, each of which takes one of two forms:*

1. *$\mu/\nu : i$, where μ and ν are prefixes (from Π) and i is the index of a modal operator (i.e., an integer in the range $[1..k]$), indicating that the transition from world μ to world ν is in the i-th accessibility relation;*

2. *$t : \mu$, where t is a first-order parameter (from L) and μ is a prefix (from Π), indicating that the parameter t is in the domain of the world μ.*

We say Σ is a typing for *a set or multiset of* signed prefixed formulas Γ *whenever Σ is a* finite *set that contains an expression $t : \mu$ for each first-order parameter t that occurs in Γ and Σ contains an expression $\mu/\nu : i$ for each prefix ν other than the empty prefix that occurs in Γ or Σ.*

The need for syntactic records of this kind only becomes evident with multi-modal logic. With only a single kind of modality, side conditions on modal rules need only refer to whether or not a transition already appears on a tableau branch. Now we must not only find the transition on the branch, but make sure that the transition is of the right kind. Recording syntactic expressions to make this determination is the most natural move. It allows us to introduce a judgment $S, \Sigma \triangleright \mu/\nu : i$ by which we indicate that the premises in Σ, together with the constraints on modal operators declared in the modal regime S, together ensure that the transition from world μ to world ν is a transition in the i-th accessibility relation. The differences among modal logics now translate into different rules for deriving the judgment $S, \Sigma \triangleright \mu/\nu : i$, in a uniform way—or what Massacci has called a “modular” way [Massacci, 1994, Massacci, 1998b]. (In modal logic, I prefer to reserve *modularity* to describe information-flow in proofs; see [Stone, 1999a].)

Definition 15 (Typings) *Assume that Σ is a typing over a language $L(CONST \cup P)^{\Pi(\kappa)}$—where in particular a base language $L(CONST)$ has been extended by first-order parameters P for the purposes of proof. Then the set of* derivable typing judgments *from Σ with respect to a regime $S = \langle A, N, Q \rangle$ is the smallest set including the expressions defined by the following conditions.*

- *(K). $S, \Sigma \triangleright \mu/\nu : i$ if $\mu/\nu : i \in \Sigma$.*
- *(T). $S, \Sigma \triangleright \mu/\mu : i$ if $A(i)$ is T, B, S4 or S5, and μ occurs in Σ.*
- *(4). $S, \Sigma \triangleright \mu/\nu : i$ if $\mu/\mu' : i \in \Sigma$, $S, \Sigma \triangleright \mu'/\nu : i$, and $A(i)$ is K4, K45, KD4, KD45, S4 or S5.*
- *(5). $S, \Sigma \triangleright \mu'/\nu : i$ if $\mu/\mu' : i \in \Sigma$, $S, \Sigma \triangleright \mu/\nu : i$, and $A(i)$ is K5, K45, KD5, KD45 or S5.*
- *(B). $S, \Sigma \triangleright \nu/\mu : i$ if $S, \Sigma \triangleright \mu/\nu : i$ and $A(i)$ is KB, KDB or B.*

- (Inc). *$S,\Sigma \triangleright \mu/\nu : j$ if $S,\Sigma \triangleright \mu/\nu : i$ and $i \leq j$ according to N .*
- (V). *$S,\Sigma \triangleright t : \mu$ if $t : \mu \in \Sigma$.*
- (I). *$S,\Sigma \triangleright t : \nu$ if Q is increasing, $S,\Sigma \triangleright \mu/\nu : i$ for some i and $S,\Sigma \triangleright t : \mu$.*
- (C). *$S,\Sigma \triangleright t : \nu$ if ν occurs in Σ and either Q is constant and $t : \mu \in \Sigma$ or t is a constant symbol ($t \in CONST$).*

We also say that there is a typing derivation $S,\Sigma \triangleright J$, or simply that $S,\Sigma \triangleright J$, when $S,\Sigma \triangleright J$ is a derivable judgment. We can treat the derivation of a typing judgment as a syntactic object, and introduce the *height* of such derivations—the number of nested rule-applications in the derivation—as a measure to perform induction on these derivations. For example, such an induction establishes (as is clear from inspection of these rules) that $S,\Sigma \triangleright \mu/\nu : i$ only if ν and μ occur in Σ.
Remark. In contrast to Massacci's rules, the inference rules of Definition 15 freely access compound transitions. With a single modal operator, it is possible to recast the inferences of Definition 15 so that any recursive rule checks for a single-step transition $\mu/\mu\alpha$ drawn from Σ. In the multi-modal case, the connections among operators prevents this in general. For example, consider the 2-modal regime S defined by $\langle A = \{0 \mapsto S4, 1 \mapsto KB\}, N = \{0 \leq 1\}, Q = constant\rangle$, and a typing $\Sigma = \{\varepsilon/\alpha : 0, \alpha/\alpha\beta : 0\}$. In this case, we must have $S,\Sigma \triangleright \alpha\beta/\varepsilon : 1$. We derive first $\varepsilon/\alpha\beta : 0$ by (4), then $\varepsilon/\alpha\beta : 1$ by (Inc), and finally $\alpha\beta/\varepsilon : 1$ by (B). However, no "single-step" derivation is possible, because there is no way to derive that $\alpha\beta/\varepsilon : 1$ where the (B) inference accesses only atomic transitions from Σ. ■

We now describe first the constituents of deductions, and then the deductions themselves. Our notation and definitions are modeled on [Goré, 1999]; we first introduce the formalism for tableau rules, and then describe the motivation for these rules—particularly the distinctive features of *modal* tableau rules.

Definition 16 (Tableau line) *A* first-order multi-modal prefixed tableau line *is an expression of the form $\Sigma \triangleright \Gamma$, where Γ is a finite multiset of signed prefixed formulas and Σ is a typing for Γ. A symbol n is* new *to a tableau line $\Sigma \triangleright \Gamma$ if there is no occurrence of n in Σ.*

Definition 17 (Tableau rule) *A* tableau rule *consists of a* numerator *L above the line and a (finite) list of* denominators *$D_1, \ldots D_n$ below the line, perhaps accompanied by a* side condition *governing the applicability of the rule. Both the numerators and the denominators are tableau lines.*

For first-order multi-modal logic over a regime S, we will use the following tableau rules:

1. *closure, with A an atomic formula:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}A^{\mu}, \mathbf{f}A^{\mu}}{\bot} \qquad \frac{\Sigma \triangleright \Gamma, \mathbf{f}\top^{\mu}}{\bot}$$

2. *conjunctive:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}A \wedge B^{\mu}}{\Sigma \triangleright \Gamma, \mathbf{t}A \wedge B^{\mu}, \mathbf{t}A^{\mu}, \mathbf{t}B^{\mu}} \qquad \frac{\Sigma \triangleright \Gamma, \mathbf{f}A \vee B^{\mu}}{\Sigma \triangleright \Gamma, \mathbf{f}A \vee B^{\mu}, \mathbf{f}A^{\mu}, \mathbf{f}B^{\mu}}$$

3. *disjunctive:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{f}A \wedge B^{\mu}}{\Sigma \triangleright \Gamma, \mathbf{f}A \wedge B^{\mu}, \mathbf{f}A^{\mu} \qquad \Sigma \triangleright \Gamma, \mathbf{f}A \wedge B^{\mu}, \mathbf{f}B^{\mu}}$$

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}}{\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}, \mathbf{t}A^{\mu} \qquad \Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}, \mathbf{t}B^{\mu}}$$

4. *negation:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}}{\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}, \mathbf{f}A^{\mu}} \qquad \frac{\Sigma \triangleright \Gamma, \mathbf{f}\neg A^{\mu}}{\Sigma \triangleright \Gamma, \mathbf{f}\neg A^{\mu}, \mathbf{t}A^{\mu}}$$

5. *possibility, subject to the side condition that* α *is new to* Σ*:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{f}\Box_i A^{\mu}}{\Sigma, \mu/\mu\alpha : i \triangleright \Gamma, \mathbf{f}\Box_i A^{\mu}, \mathbf{f}A^{\mu\alpha}} \qquad \frac{\Sigma \triangleright \Gamma, \mathbf{t}\Diamond_i A^{\mu}}{\Sigma, \mu/\mu\alpha : i \triangleright \Gamma, \mathbf{t}\Diamond_i A^{\mu}, \mathbf{t}A^{\mu\alpha}}$$

6. *necessity—subject to the side condition that there is a typing derivation* $S, \Sigma \triangleright \mu/\nu : i$*:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}\Box_i A^{\mu}}{\Sigma \triangleright \Gamma, \mathbf{t}\Box_i A^{\mu}, \mathbf{t}A^{\nu}} \qquad \frac{\Sigma \triangleright \Gamma, \mathbf{f}\Diamond_i A^{\mu}}{\Sigma \triangleright \Gamma, \mathbf{f}\Diamond_i A^{\mu}, \mathbf{f}A^{\nu}}$$

7. *special necessity—subject to the side condition that* $A(i)$ *is one of KD, KDB, KD4, KD5 or KD45, that* $i \leq j$ *according to* N *and that* α *is a modal parameter new to* Σ*:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}\Box_j A^{\mu}}{\Sigma, \mu/\mu\alpha : i \triangleright \Gamma, \mathbf{t}\Box_j A^{\mu}, \mathbf{t}A^{\mu\alpha}} \qquad \frac{\Sigma \triangleright \Gamma, \mathbf{f}\Diamond_j A^{\mu}}{\Sigma, \mu/\mu\alpha : i \triangleright \Gamma, \mathbf{f}\Diamond_j A^{\mu}, \mathbf{f}A^{\mu\alpha}}$$

8. *extra special necessity—subject to the side conditions that* $A(i)$ *is one of KD, KDB, KD4, KD5 or KD45, that* $A(j)$ *is one of K5, K45, KD5, KD45 or S5, that* $i \leq j$ *according to* N*, that* $S, \Sigma \triangleright \mu/\nu : j$*, and that* α *is a modal parameter new to* Σ*:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{u}A^{\nu}}{\Sigma, \mu/\mu\alpha : i \triangleright \Gamma, \mathbf{t}\top^{\mu\alpha}, \mathbf{u}A^{\nu}}$$

9. *existential—subject to the side condition that c is a first-order parameter new to* Σ*:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}\exists x A^{\mu}}{\Sigma, c : \mu \triangleright \Gamma, \mathbf{t}\exists x A^{\mu}, \mathbf{t}A[c/x]^{\mu}} \qquad \frac{\Sigma \triangleright \Gamma, \mathbf{f}\forall x A^{\mu}}{\Sigma, c : \mu \triangleright \Gamma, \mathbf{f}\forall x A^{\mu}, \mathbf{f}A[c/x]^{\mu}}$$

10. *universal—subject to the side condition that there is a typing derivation* $S, \Sigma \triangleright t : \mu$*:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}\forall x A^{\mu}}{\Sigma \triangleright \Gamma, \mathbf{t}\forall x A^{\mu}, \mathbf{t}A[t/x]^{\mu}} \quad \frac{\Sigma \triangleright \Gamma, \mathbf{f}\exists x A^{\mu}}{\Sigma \triangleright \Gamma, \mathbf{f}\exists x A^{\mu}, \mathbf{f}A[t/x]^{\mu}}$$

Definition 18 (Tableau) *A* first-order multi-modal tableau *for X (over regime S) is a finite tree in which each node carries a* tableau line*, and in particular the root carries the line X, such that when an internal node carries line Y and its children carry lines $Z_1, \ldots, Z_n$, $Y/Z_1 \ldots Z_n$ instantiates a tableau rule over S in such a way as any side conditions on the tableau rule are met.*

Remark (Rules). A tableau is a syntactic record that describes a systematic search for a model in which all formulas from Γ are satisfied. Such a search fails at a closure inference, for no model can simultaneously make the same atomic formula both true and false at the same world. Meanwhile, each inference rule decomposes some formula according to its outermost logical connective, so as to explore the different ways a model could satisfy that formula. For example, the conjunctive figure

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}A \wedge B^{\mu}}{\Sigma \triangleright \Gamma, \mathbf{t}A \wedge B^{\mu}, \mathbf{t}A^{\mu}, \mathbf{t}B^{\mu}}$$

indicates that, since both A and B must be true at world μ for $A \wedge B$ to be true at μ, search for a model for $A \wedge B$ succeeds only when both A and B are satisfied at μ. Conversely, the disjunctive figure

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}}{\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}, \mathbf{t}A^{\mu} \qquad \Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}, \mathbf{t}B^{\mu}}$$

indicates that, since $A \vee B$ is true at a world μ as long as either A is true at μ or B is, search for a model to $A \vee B$ may involves finding a structure which is a model of A, or it may involve finding one which is a model of B.

A tableau *proof*—a tableau in which every path eventually reaches a closure inference—then indicates that all possible ways of constructing a model have been tried, and have failed. If Γ has the form $\mathbf{f}E$, the tableau then shows that there is no way to make E false: E must hold in all models.

As with the boolean rules, tableau rules for modal operators and quantifiers can be viewed as a function of the semantics of modal formulas and quantified formulas. In the case of the possibilty and necessity rules, this view is relatively straightforward. For possibility, for example, a model for $\Diamond_i A$ at μ must satisfy A at some world accessible from μ. The inference rule gives this world an arbitrary name $\mu\alpha$, and continues the search for a model. At necessity, meanwhile, a model which makes $\Box_i A$ true at a world μ must have A true at any accessible world ν in the model; the inference rule checks that ν is acessible from μ, and continues the search for the model assuming also that A is true at ν.

Clearly, however, the proof-theoretic treatment of serial modalities—as represented in the special necessity and extra special necessity inference figures—involves additional complexity. From the point of view of uniformity, the simplest treatment of serial modalities would be to introduce a general rule of the following form—

$$\frac{\Sigma \triangleright \Gamma}{\Sigma, \mu/\mu\alpha : i \triangleright \Gamma}$$

—subject to the side condition that μ occurs in Σ and $A(i)$ is one of KD, KDB, KD4, KD5 or KD45. (In fact, since the domains of quantifiers cannot be empty, a similar rule would be needed to infer $\exists xA$ from $\forall xA$—except that since we always have some constant symbol that we can instantiate the universal statement to, we do not need a special rule to introduce a fresh symbol for this purpose.) Observe that the extra special necessity rule is just an instance of this rule, except for the addition of an always true signed prefixed formula in the denominator of the tableau rule. The special necessity rule, meanwhile, combines an application of this rule with an application of the necessity rule to instantiate a formula at the new prefix $\mu\alpha$.

There are a number of advantages to the use of the special necessity rule. The special necessity rule encodes the fact that formulas cannot probe paths of accessibility whose length exceeds the modal depth of the formula, an important semantic generalization about modal logic, cf. [van Benthem, 1983]. At the same time, the rule makes for a tighter proof system about which stronger results can be proved. For example, the special necessity rule gives easy decision procedures for combinations of KD modal operators in the propositional case: with it, tableaux can only introduce prefixes whose length equals the modal depth of the formulas being proved. In multi-modal regimes where the special necessity rule suffices—regimes where serial operators never narrow euclidean operators—it makes sense to use it. That motivates its adoption here, where a major goal—as suggested in section 1 and underscored in section 5—is to lay the groundwork for computational investigation of particular modal theories and fragments.

Unfortunately, the extra special necessity rule *is* indispensible when we *do* have serial operators that narrow euclidean operators. In this case there is no local test that permits us to determine when we may have to look at the witness world for a serial modality in order to derive a contradiction. An example will give the flavor of the difficulty. Consider this regime

$$S = \langle A = \{1 \mapsto KD, 2 \mapsto K5, 3 \mapsto K5\}, N = \{1 \leq 2, 1 \leq 3\}, Q = constant\rangle$$

Then we have a closed tableau

$$
\begin{array}{c|l}
\rhd \mathbf{t}\Diamond_2\Box_2 A,\ \mathbf{t}\Diamond_3\Box_3\neg A & \\
\hline
/\alpha:2 \rhd \mathbf{t}\Box_2 A^{\alpha},\ \mathbf{t}\Diamond_3\Box_3\neg A & \mathbf{t}\Diamond_2 \\
\hline
/\alpha:2,\ /\beta:3 \rhd \mathbf{t}\Box_2 A^{\alpha},\ \mathbf{t}\Box_3\neg A^{\beta} & \mathbf{t}\Diamond_3 \\
\hline
/\alpha:2,\ /\beta:3,\ \gamma:1 \rhd \mathbf{t}\Box_2 A^{\alpha},\ \mathbf{t}\Box_3\neg A^{\beta},\ \mathbf{t}\top^{\gamma} & \text{extra special} \\
\hline
/\alpha:2,\ /\beta:3,\ \gamma:1 \rhd \mathbf{t}A^{\gamma},\ \mathbf{t}\Box_3\neg A^{\beta},\ \mathbf{t}\top^{\gamma} & \mathbf{t}\Box_2 \\
\hline
/\alpha:2,\ /\beta:3,\ \gamma:1 \rhd \mathbf{t}A^{\gamma},\ \mathbf{t}\neg A^{\gamma},\ \mathbf{t}\top^{\gamma} & \mathbf{t}\Box_3 \\
\hline
\bot & \text{closure}
\end{array}
$$

In displaying this proof, the preservation of principal formulas of tableau rules is suppressed for readability. The extra special necessity rule applies at the third step to introduce world γ. Thereafter, we can derive $\alpha/\gamma:2$ and $\alpha/\gamma:3$ by (*Inc*) and (5); instantiating the necessary formulas there derives a contradiction.

There is a final feature that makes the special necessity rule attractive, and which motivates the difference between the tableau scheme given above and the extra special necessity rule. This is that they allow the worlds relevant to satisfying a tableau line $\Sigma \rhd \Gamma$ to be determined from the prefixes that occur in Γ. This permits shortcuts in the metatheory, particular in the completeness proof. ■

Remark (Structure). Each of the tableau rules of Definition 17 (except closure) applies at a node in virtue of the presence of a distinguished signed prefixed formula that the node carries; we refer to this as the *principal expression* or simply the *principal* of the rule or rule application. (One typically refers to the principal formula of a rule; but in this report the objects in question are not simply formulas, but expressions built out of formulas, as well as prefixes, signs, etc.) Similarly, each of the tableau rules introduces new expressions onto the branch which we refer to as the *side expressions* of the rule. Typically, the side expressions of a rule are constituents (modulo instantiation of terms for variables and a change in prefix) of the rule's principal expression—the exception being the extra special necessity rule (where the side formula serves, as noted above, only to register a new prefix on the tableau line).

A *branch* of a tableau is a path in the tableau from the root to a leaf. Tableau rules are often written so that each node carries just the new formulas introduced there. On this convention, the closure conditions must check that a formula and its complement occur on the same branch of the tableau. A tableau so written may be converted to a tableau using the figures of Definition 17 by relabling each node to include all formulas higher on any branch through the node. The tableau rules of Definition 17 bear an evident resemblance to the rules of the sequent calculus. Suppose we break down a multiset of signed prefixed formulas Γ into a multiset of formulas Ψ signed $\mathbf{t}$ and a multiset of formulas Δ signed $\mathbf{f}$. Then Γ corresponds to the sequent $\Psi \longrightarrow \Delta$ and now the tableau figures of Definition 17 may be regarded just as sequent calculus rules written upside-down. In particular, the rules follow a convention of dispensing with the *structural rules* of the sequent calculus—avoiding the weakening rule by permitting a multiset Γ of "extraneous" formulas on the closure rule; and avoiding the contraction rule by automatically preserving all formulas at the application of logical rules. Finally, we remark that the use of *signed* expressions means that each rule comes in double; we will refer to an application of the rule to an expression signed **t** as a *positive* one and an application to an expression signed **f** as a *negative* one. ■

A branch is *closed* just in case the leaf on the branch carries the label $\bot$. A tableau is *closed* just in case every branch of the tableau is closed.

Definition 19 (Provability) *Let Γ be a set of prefixed formulas and A be a prefixed formula, and let Σ be set of typing expressions. Then A follows from Γ under Σ—written $\Sigma \triangleright \Gamma \longrightarrow A$ just in case there is a finite multiset $\Gamma_0 \subset \Gamma$ and a finite subset $\Sigma_0 \subset \Sigma$ such that Σ_0 is a typing for Γ_0, A and there is a closed tableau for $\Sigma_0 \triangleright \Gamma_0, \mathbf{f}A$.*

In proving properties of tableaux and tableau proofs, we have two structural strategies. The first is based on the natural notion of subproof. Given a tableau—consisting of a tree T whose nodes carry tableau lines—then any subtree of T' (with the same nodes carrying the same lines) is also a tableau; we can call T' a subtableau (or if T is a proof, a subproof) of of T.

The second method is based on viewing tableaux as composed by *branch extension*. Two paths (or branches) *agree* if they contain the same number of nodes and corresponding nodes carry identical labels; branch b' extends branch b if b and the path obtained by removing the leaf from b' agree. In general, if the leaf of a branch carries the line $\Sigma \triangleright \Gamma$, we say the branch *ends in* $\Sigma \triangleright \Gamma$.

Then the tableau T' *extends* the tableau T if any branch of T but one, b, agrees with some branch of T', and every branch of T' agrees with a branch of T or extends b.

2.4 Soundness

We begin by showing that the proof system is sound: that a formula is never provable unless it is true in all models; we adapt the arguments presented in [Fitting, 1983, 2.3] and [Fitting and Mendelsohn, 1998, 5.3]. These arguments formalize the intuitive motivation for the tableau system as providing a systematic search for a model for formulas. To be precise about the kind of model that a tableau should construct, we first define satisfiability in a model with respect to an assignment. The main lemma, Lemma 4, then shows that the search for these models involved in the application of the tableau rules leaves open all possibilities. So, if a tableau line has a model, there is no way for a tableau proof to report that there is no model for it.

Definition 20 (Satisfiable) *Given a regime S, suppose Γ is a set of signed prefixed formulas (over $L(CONST \cup P)^{\Pi(\kappa)}$) and Σ is a set of typing expressions in the same* *language. We say* Γ *is* satisfiable *against* Σ *in a model* $\mathcal{M} = \langle G, \mathcal{R}, D, J\rangle$ *that respects* S *with respect to an assignment* g *if there is a map* θ *assigning each prefix* μ *that occurs in* Σ *to a world* $\theta(\mu) \in G$ *such that:*

1. *For any expression* $\mu/\nu : i$ *in* Σ, $\theta(\mu)\mathcal{R}_i\theta(\nu)$.
2. *For any parameter* p *with an expression* $p : \mu$ *in* Σ, $g(p) \in D(\theta(\mu))$.
3. *For any signed prefixed formula* $\mathbf{t}A^{\mu} \in \Gamma$, $\mathcal{M}, \theta(\mu) \Vdash_g A$, *and for any signed prefixed formula* $\mathbf{f}A^{\mu} \in \Gamma$, $\mathcal{M}, \theta(\mu) \nVdash_g A$.

Once we find an assignment and function that match Σ (according to conditions 1 and 2 of Definition 20), we thereby ensure respect for all the typing judgments derivable from Σ according to the regime S.

Lemma 2 *Let* Σ *be a set of typing expressions, let* $\mathcal{M} = \langle G, \mathcal{R}, D, J\rangle$ *be a model that respects the regime* $S = \langle A, \mathcal{N}, \mathcal{Q}\rangle$*, and let* θ *be a map from the prefixes that occur in* Σ *to worlds in* G *such that for all expressions* $\mu'/\nu' : i$ *in* Σ, $\theta(\mu')\mathcal{R}_i\theta(\nu')$. *Then if* $S, \Sigma \triangleright \mu/\nu : i$ *for prefixes* μ *and* ν *that occur in* Σ *then* $\theta(\mu)\mathcal{R}_i\theta(\nu)$.

Proof. By induction on the height of the derivation of $\Sigma \triangleright \mu/\nu : i$.

The base cases are as follows:

1. $\mu/\nu : i$ in Σ (K)—so the result follows by assumption;
2. $\mu = \nu$ and $A(i)$ is T, B, S4 or S5 (T) and μ occurs in Σ—so the result follows because θ must be defined on μ and hence $\theta(\mu) = \theta(\nu)$ and $\mathcal{R}_i$ is reflexive.

Suppose the induction true of derivations of height $h-1$ or less and consider a derivation of height h according to the clause which introduces it:

1. (B): we get $\mu/\nu : i$ by showing $S, \Sigma \triangleright \nu/\mu : i$ in fewer steps with $A(i)$ one of KB, KDB or B—so the result follows because by assumption $\theta(\nu)\mathcal{R}_i\theta(\mu)$ and $\mathcal{R}_i$ is symmetric
2. (4): we get $\mu/\nu : i$ by $\mu/\mu' \in \Sigma$ and a shorter derivation of $S, \Sigma \triangleright \mu'/\nu : i$. By IH then $\theta(\mu)\mathcal{R}_i\theta(\mu')$ and $\theta(\mu')\mathcal{R}_i\theta(\nu)$. Since $A(i)$ is K4, K45, KD4, KD45, S4 or S5, $\mathcal{R}_i$ is transitive, and $\theta(\mu)\mathcal{R}_i\theta(\nu)$
3. (5): we get $\mu'/\nu : i$ by $S, \Sigma \triangleright \mu/\mu' : i$ and $S, \Sigma \triangleright \mu/\nu : i$. By IH, $\theta(\mu)\mathcal{R}_i\theta(\mu')$ and $\theta(\mu)\mathcal{R}_i\theta(\nu)$. Since $A(i)$ is K5, K45, KD5, KD45 or S5, $\mathcal{R}_i$ is euclidean and hence $\theta(\mu\alpha)\mathcal{R}_i\theta(\nu)$.
4. (Inc): we get $\mu/\nu : j$ by showing $S, \Sigma \triangleright \mu/\nu : i$ with $i \leq j$ according to $\mathcal{N}$. But by IH, $\theta(\mu)\mathcal{R}_i\theta(\nu)$ and since the frame is narrowing from j to i, $\theta(\mu)\mathcal{R}_j\theta(\nu)$.

This establishes the result. ■

Not only the modal parameters but also the first-order parameters are treated properly.

Lemma 3 *Let Σ be a set of typing expressions, let $M = \langle G, R, D, J \rangle$ be a model that respects the regime $S = \langle A, N, Q \rangle$, let θ be a map from prefixes that occur in Σ to worlds and let g be an assignment such that for any expression $\mu/\nu : i \in \Sigma$ then $\theta(\mu) R_i \theta(\nu)$ and for any parameter p with an expression $p : \mu \in \Sigma$, $g(p) \in D(\theta(\mu))$. Then if $S, \Sigma \triangleright p : \nu$ then $g(p) \in D(\theta(\nu))$*

Proof. The judgment $S, \Sigma \triangleright p : \nu$ is derived in one of three ways, depending in part on Q. In the base case, the judgment may be derived from $p : \nu \in \Sigma$, but then $g(p) \in D(\theta(\nu))$ by assumption. Or, if Q is constant, then we know $p : \mu \in \Sigma$ and ν also occurs in Σ. Then $\theta(\nu)$ names some possible world, and $g(p) \in D(\theta(\mu))$. But since this is a constant domain model, $D(\theta(\mu)) = D(\theta(\nu))$.

Finally, inductively, if Q is increasing, the judgment may be derived from a shorter derivation of $S, \Sigma \triangleright p : \mu$ and a derivation that $S, \Sigma \triangleright \mu/\nu : i$. By Lemma 2, $\theta(\mu) R_i \theta(\nu)$. But then since M respects the regime, $D(\theta(\mu)) \subseteq D(\theta(\nu))$. Since by hypothesis $g(p) \in D(\theta(\mu))$, $g(p) \in D(\theta(\nu))$. ■

A tableau branch is *satisfiable* it ends in $\Sigma \triangleright \Gamma$—and Γ is satisfiable against Σ in some model with respect to some assignment. A tableau is satisfiable if some branch in the tableau is satisfiable.

Lemma 4 (Extension) *Any extension T' of a satisfiable tableau T is satisfiable.*

Proof. We suppose T is satisfiable. By definition, for all the branches of T but one, b, there is a branch of T' that agrees. Hence, we need only consider the case where b is the unique satisfiable branch of T; we can refer to the model $M = \langle G, R, D, J \rangle$, the world function θ and the assignment g which witness the satisfiability of b. In T', the path b occurs; but now it ends in an internal node which instantiates some tableau rule. We show that T' is satisfiable by case analysis on this rule.

If the rule is *closure*, there is one branch b' in T' extending b, and since the leaf on b' is $\perp$, b' is not satisfiable by definition. But for closure to apply, b must end either in $\Sigma \triangleright \Gamma, \mathbf{t}A^\mu, \mathbf{f}A^\mu$ or in $\Sigma \triangleright \Gamma, \mathbf{f}\top^\mu$. In the first case we must have both $M, \theta(\mu) \Vdash_g A$ and $M, \theta(\mu) \nVdash_g A$. This is absurd. The second case is likewise impossible, as it requires $M, \theta(\mu) \nVdash_g \top$.

If the rule is *conjunctive*, the following reasoning for the *positive* instance is representative. There is one branch b' in T' extending b; the leaf of b must carry $\Sigma \triangleright \Gamma, \mathbf{t}A \wedge B^\mu$ while the leaf of b' must carry $\Sigma \triangleright \Gamma, \mathbf{t}A \wedge B^\mu, \mathbf{t}A^\mu, \mathbf{t}B^\mu$. To show the satisfiabilty of b' it suffices to show $M, \theta(\mu) \Vdash_g A$ and $M, \theta(\mu) \Vdash_g B$. But this follows from the fact that $M, \theta(\mu) \Vdash_g A \wedge B$—which we know from the satisfiability of b—and the definition of truth in a model.

If the rule is *negation*, the same reasoning applies *mutatis mutandis*.

If the rule is *disjunctive*, the following reasoning for the *positive* instance is representative. There are two branches b' and b'' in $\mathcal{T}'$ extending b. The leaf of b must carry $\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}$ while let us say the leaf of b' carries $\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}, \mathbf{t}A^{\mu}$, while that of b'' carries $\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}, \mathbf{t}B^{\mu}$. To show *one* of b' and b'' satisfiable, it suffices to show that *either* $\mathcal{M}, \theta(\mu) \Vdash_g A$ or $\mathcal{M}, \theta(\mu) \Vdash_g B$. Again, this follows from the fact that $\mathcal{M}, \theta(\mu) \Vdash_g A \vee B$—which we know from the satisfiability of b—and the definition of truth in a model.

If the rule is *possibility*, the following reasoning for the *positive* instance is representative. There is one branch b' in $\mathcal{T}'$ extending b; the leaf of b must carry $\Sigma \triangleright \Gamma, \mathbf{t}\Diamond_i A^{\mu}$ while the leaf of b' must carry $\Sigma, \mu/\mu\alpha : i \triangleright \Gamma, \mathbf{t}\Diamond_i A^{\mu}, \mathbf{t}A^{\mu\alpha}$ for α new. Now we know $\mathcal{M}, \theta(\mu) \Vdash_g \Diamond_i A$, so there must be a world w such that $\theta(\mu) \mathcal{R}_i w$ and $\mathcal{M}, w \Vdash_g A$ — by the definition of truth in a model. So construct a function θ' exactly like θ except that $\theta'(\mu\alpha) = w$. For $\mu/\mu\alpha : i$ then, $\theta'(\mu) \mathcal{R}_i \theta'(\mu\alpha)$. Moreover, since θ' coincides with θ on the prefixes of Σ and Γ, for any B^{ν} in Γ, $\mathcal{M}, \theta'(\mu) \Vdash_g B$ if and only if $\mathcal{M}, \theta'(\mu) \Vdash_g B$. For the same reason θ' meets conditions 1 and 2 of Definition 20 (on Σ, with respect to g). It follows that $\Gamma, \mathbf{t}\Diamond_i A^{\mu}, \mathbf{t}A^{\mu\alpha}$ is satisfiable against $\Sigma, \mu/\mu\alpha : i$ in $\mathcal{M}$ with respect to g and θ'.

If the rule is *necessity*, again we consider a representative positive case. There is one branch b' in $\mathcal{T}'$ extending b; the leaf of b must carry $\Sigma \triangleright \Gamma, \mathbf{t}\Box_i A^{\mu}$ while the leaf of b' must carry $\Sigma \triangleright \Gamma, \mathbf{t}\Box_i A^{\mu}, \mathbf{t}A^{\nu}$ with $\Sigma \triangleright \mu/\nu : i$. Since $\Sigma \triangleright \mu/\nu : i$ by Lemma 2, $\theta(\mu) \mathcal{R}_i \theta(\nu)$. Moreover, we have $\mathcal{M}, \theta(\mu) \Vdash_g \Box_i A$. But then by the definition of truth, $\mathcal{M}, \theta(\nu) \Vdash_g A$. So b' is satisfiable.

If the rule is *special necessity*, we combine the preceding arguments. We consider a new world w provided by the seriality of modality i and construct a variant θ' of θ with $\theta'(\mu\alpha) = w$. The argumentation from the possibilistic case allows us to reduce the satisfiability of the branch with respect to θ' to some condition $\mathcal{M}, w \Vdash_g A$ (in a positive rule); this follows from the necessity of $\Box_j A$ as in the necessity case, thanks to a derivation of $\mu/\mu\alpha : j$ from $\mu/\mu\alpha : i$ using the fact that $i \leq j$ according to $\mathcal{N}$.

If the rule is *extra special necessity*, it suffices simply to consider a new world w provided by the seriality of modality i and construct a variant θ' of θ with $\theta'(\mu\alpha) = w$.

If the rule is *existential*, again we consider a representative positive case. There is one branch b' in $\mathcal{T}'$ extending b; the leaf of b must carry $\Sigma \triangleright \Gamma, \mathbf{t}\exists x A^{\mu}$ while the leaf of b' must carry $\Sigma, c : \mu \triangleright \Gamma, \mathbf{t}\exists x A^{\mu}, \mathbf{t}A[c/x]^{\mu}$, for some new c. Now we know $\mathcal{M}, \theta(\mu) \Vdash_g \exists x A$ so by the definition of truth in a model there must be some individual $u \in \mathcal{D}(\theta(\mu))$ such that $\mathcal{M}, \theta(\mu) \Vdash_{g'} A$ for an x-variant g' of g with $g'(x) = u$. Now we consider an assignment g'' which is in fact a c-variant of g for the parameter c, with $g''(x) = c$. It is immediate that g'' satisfies conditions 1 and 2 of Definition 20. Moreover, since c does not occur in Σ or Γ, we can apply Lemma 1—for any variable z that does not occur in Γ—to show of any B^{μ} in Γ that $\mathcal{M}, \theta \Vdash_{g''} B$; it remains only to show $\mathcal{M}, \theta(\mu) \Vdash_{g''} A[c/x]$. But this follows by another application of Lemma 1 with x and c.

If the rule is *universal*, we can reason as with the following positive case. There is one branch b' in $\mathcal{T}'$ extending b; the leaf of b must carry $\Sigma \triangleright \Gamma, \mathbf{t}\forall x A^{\mu}$ while the leaf of b' must carry $\Sigma \triangleright \Gamma, \mathbf{t}\forall x A^{\mu}, A[t/x]^{\mu}$, for some t with $\Sigma \triangleright t : \mu$. Again we know that $\mathcal{M}, \theta(\mu) \Vdash_g \forall x A$. Let $u = g(t)$ if t is a parameter; we know by Lemma 3, that $u \in D(\theta(\mu))$. Otherwise, let $u = J(t)$ (if t is a constant); because $J(t) \in C(\mathcal{F})$, $u \in D(\theta(\mu))$. In either case, by the definition of truth, $\mathcal{M}, \theta(\mu) \Vdash'_g A$ with g' and x-variant of g with $g'(x) = u$. Then, again by Lemma 1, $\mathcal{M}, \theta(\mu) \Vdash_g A[t/x]$, and so b' is satisfiable. ■

Theorem 1 (Soundness) *Suppose there is a closed tableau $\mathcal{T}$ for $\triangleright \mathbf{f}A^{\varepsilon}$. Then A is valid.*

Proof. By contradiction: suppose A is not valid. Then there is some model $\mathcal{M}$, world w and assignment g such that $\mathcal{M}, w \Vdash_g \neg A$. This means that the tableau $\mathcal{T}_0$ consisting of a single node carrying $\triangleright \neg A$ is satisfiable. Hence, by the lemma, so is any tableau we get from $\mathcal{T}_0$ and applying branch extension rules—in particular $\mathcal{T}$ is satisfiable. But $\mathcal{T}$ cannot be satisfiable, since $\mathcal{T}$ is closed. Thus A must be valid. ■

2.5 Completeness

We now turn to the completeness theorem, which states that if a formula is valid then there is a proof for it. In fact, we prove the contrapositive: if there is no proof for the formula, then there is a model where the formula is false. Again, the argument behind the completeness theorem can be seen as a formalization of the motivation for tableaux in systematic search for models. In this case, the idea is that this systematic search, if carried far enough, will construct a countermodel to a formula if a countermodel exists. Otherwise, the search must fail, giving a syntactic proof for the formula. Now, modal formulas may be satisfied only in infinite models, so the completeness theorem effectively requires us to consider infinite sequences of applications of tableau rules. In moving to infinite sets in this way, we must formally move from tableaux, viewed as syntactic objects, to a more abstract, algebraic characterization of sets of modal formulas. In particular, we will follow [Fitting, 1983] in developing the completeness argument in terms of *analytic consistency properties* for the modal language. The bridge from finite tableaux to infinite consistency properties is mediated by an extended construction—presented in section 2.5.1 with the runup to Proposition 8—that develops a characterization for an infinite set of formulas in terms of the behavior, on finite subsets of that set, of rewrite rules like those used in the construction of a tableau.

In this setting, the systematic, *infinite* application of tableau rules corresponds to finding a fixed point for the rewrite rules. This is formalized in section 2.5.2 with the notion of a downward saturated set: a set which already contains all the formulas that might possibly be added along a hypothetical infinite branch in a tableau. We show there how to start with the kinds of sets for which tableau search fails to derive a closed tableau (characterized algebraically in terms of analytic consistency properties) and extend them into downward saturated sets—with some care about the introduction of representative possible worlds and first-order parameters to witness possible and existential statements. It is a short step from a downward saturated set to a corresponding modal model.

The proof of completeness itself is then presented in section 2.5.3. It links the results together, showing how to conclude that there is a countermodel for a formula as long as there is no corresponding closed tableau. Because we are reasoning in an expressive logical language using a proof system without cut, the development of this completeness proof shoulders a relatively large burden. Much simpler completeness proofs are possible for proof systems with cut, but they then require a syntactic proof of cut-elimination to derive a computational proof system for automated reasoning.

2.5.1 Consistency Properties

Definition 21 (Consistency property) *Given a regime* $S = \langle A, N, Q \rangle$*, let* Σ *be a typing over a language* $L(CONST \cup P)^{\Pi(\kappa)}$ *and let* $\mathbf{C}$ *be a collection of sets of signed prefixed formulas from* $L(CONST \cup P)^{\Pi(\kappa)}$. $\mathbf{C}$ *is a* first-order S-consistency property for Σ *if for each set* $S \in \mathbf{C}$ *the following conditions are met:*

1. *every prefix* μ *that occurs in S occurs in* Σ*; and whenever some first order parameter* $p \in P$ *occurs in S, p also occurs in* Σ*.*

2. *there is no atomic formula A and prefix* μ *with* $\mathbf{t}A^{\mu} \in S$ *and* $\mathbf{f}A^{\mu} \in S$*, nor is* $\mathbf{f}\top^{\mu} \in S$*.*

3. $\mathbf{t}A \wedge B^{\mu} \in S \Rightarrow S \cup \{\mathbf{t}A^{\mu}, \mathbf{t}B^{\mu}\} \in \mathbf{C}$*. Likewise,* $\mathbf{f}A \vee B^{\mu} \in S \Rightarrow S \cup \{\mathbf{f}A^{\mu}, \mathbf{f}B^{\mu}\} \in \mathbf{C}$.

4. $\mathbf{t}A \vee B^{\mu} \in S \Rightarrow$ *either* $S \cup \{\mathbf{t}A^{\mu}\} \in \mathbf{C}$ *or* $S \cup \{\mathbf{t}B^{\mu}\} \in \mathbf{C}$*. Likewise,* $\mathbf{f}(A \wedge B)^{\mu} \in S \Rightarrow$ *either* $S \cup \{\mathbf{f}A^{\mu}\} \in \mathbf{C}$ *or* $S \cup \{\mathbf{f}B^{\mu}\} \in \mathbf{C}$.

5. $\mathbf{t}\neg A^{\mu} \in S \Rightarrow S \cup \{\mathbf{f}A^{\mu}\} \in \mathbf{C}$*. Likewise,* $\mathbf{f}\neg A^{\mu} \in S \Rightarrow S \cup \{\mathbf{t}A^{\mu}\} \in \mathbf{C}$.

6. $\mathbf{t}\forall x A^{\mu} \in S \Rightarrow S \cup \{\mathbf{t}A[p/x]\} \in \mathbf{C}$ *for every symbol p satisfying either (1)* $p \in CONST$*, or (2)* $p \in P$*, p occurs in S, and* $S, \Sigma \triangleright p : \mu$*. Likewise,* $\mathbf{f}\exists x A^{\mu} \in S \Rightarrow S \cup \{\mathbf{f}A[p/x]\} \in \mathbf{C}$ *for every symbol p satisfying either (1)* $p \in CONST$*, or (2)* $p \in P$*, p occurs in S, and* $S, \Sigma \triangleright p : \mu$*.*

7. $\mathbf{t}\exists x A^{\mu} \in S \Rightarrow$ *there is some symbol* $p \in CONST \cup P$ *such that* $S, \Sigma \triangleright p : \mu$ *and* $S \cup \{\mathbf{t}A[p/x]\} \in \mathbf{C}$*. Likewise,* $\mathbf{f}\forall x A^{\mu} \in S \Rightarrow$ *there is some symbol* $p \in CONST \cup P$ *such that* $S, \Sigma \triangleright p : \mu$ *and* $S \cup \{\mathbf{f}A[p/x]\} \in \mathbf{C}$.

8. $\mathbf{t}\Box_i A^{\mu} \in S \Rightarrow S \cup \{\mathbf{t}A^{\nu}\} \in \mathbf{C}$ *for every prefix* $\nu \in \Pi(\kappa)$ *such that* $S, \Sigma \triangleright \mu/\nu : i$ *and there is some* $\mathbf{u}B^{\mu} \in S$*. Likewise,* $\mathbf{f}\Diamond_i A^{\mu} \in S \Rightarrow S \cup \{\mathbf{f}A^{\nu}\} \in \mathbf{C}$ *for every prefix* $\nu \in \Pi(\kappa)$ *such that* $S, \Sigma \triangleright \mu/\nu : i$ *and there is some* $\mathbf{u}B^{\mu} \in S$*.*

9. *Suppose either (1)* $\mathbf{t}\Diamond_i A^\mu \in S$ *or (2)* $\mathbf{t}\Box_j A^\mu \in S$ *and* $A(i)$ *is KD, KDB, KD4, KD5 or KD45 with* $i \leq j$ *by* N*: then* $S \cup \{\mathbf{t}A^{\mu\alpha}\} \in \mathbf{C}$ *for some prefix* $\mu\alpha \in \Pi(\kappa)$ *such that* $\mu/\mu\alpha : i \in \Sigma$*. Likewise, suppose either (1)* $\mathbf{f}\Box_i A^\mu \in S$ *or (2)* $\mathbf{f}\Diamond_j A^\mu \in S$ *and* $A(i)$ *is KD, KDB, KD4, KD5 or KD45 with* $i \leq j$ *by* N*: then* $S \cup \{\mathbf{f}A^{\mu\alpha}\} \in \mathbf{C}$ *for some prefix* $\mu\alpha \in \Pi(\kappa)$ *such that* $\mu/\mu\alpha : i \in \Sigma$*.*

10. *If* $\mathbf{u}A^\nu \in S$ *with* $S, \Sigma \triangleright \mu/\nu : j$ *for* $A(j)$ *one of K5, K45, KD5, KD45 or S5, for* $i \leq j$ *by* N *and* $A(i)$ *one of KD, KDB, KD4, KD5 or KD45, then* $S \cup \{\mathbf{t}\top^{\mu\alpha}\} \in \mathbf{C}$ *for some* α *such that* $\mu/\mu\alpha : i \in \Sigma$*.*

Let $\mathbf{C}$ *be a first-order* S*-consistency property for* Σ *and let* X *be a set of formulas from* $L(CONST)$*. We call* $\mathbf{C}$ X-compatible *if, for each* $S \in \mathbf{C}$ *and for each* $A \in X$*,* $S \cup \{\mathbf{t}A^\varepsilon\} \in \mathbf{C}$*.*

Now, in order to work with consistency properties, we need to place some sensible constraints on typings.

Definition 22 (Fairness) *Given a typing* Σ*, denote by* $\Sigma_\kappa(\mu, i)$ *the set* $\{\alpha \in \kappa \mid \mu/\mu\alpha : i \in \Sigma\}$*. Denote by* $\Sigma_P(\mu)$ $\{p \in P \mid p : \mu \in \Sigma\}$*. A typing* Σ *is* fair *to a countable set of modal parameters* κ *and a countable set of first-order parameters* P *if:*

- *for any* $\alpha \in \kappa$*, there is a unique expression* $\mu/\nu\alpha : i \in \Sigma$ *and* $\mu = \nu$*;*
- *for any* $p \in P$*, there is a unqiue expression* $p : \mu \in \Sigma$*;*
- *for any* $\mu \in \Pi(\kappa)$ *and any modality* i*,* $\Sigma_\kappa(\mu, i)$ *and* $\Sigma_P(\mu)$ *are countably infinite.*

Remark (existence). For any countably infinite sets κ and P, we can construct a fair typing. Let α_i enumerate κ and let p_i enumerate P. Meanwhile, let $\langle \mu_i, m_i \rangle_i$ be an enumeration of the pairs of prefixes of $\Pi(\kappa)$ and integers indexing modalities with infinite repetition. We can choose these enumerations such that when any element α_j occurs in μ_i, then $j < i$. Now use the typing:

$$\{\mu_i/\mu_i\alpha_i : m_i \mid \text{ for any integer } i\} \cup \{p_i : \mu_i \mid \text{ for any integer } i\}$$

Moreover, given a fair typing Σ for κ and P, we can partition κ and P into countably many disjoint countably infinite sets $\kappa_1 \ldots$ and $P_1 \ldots$ such that Σ is still fair if restricted to typing expressions for prefixes over $\bigcup_{n<m} \kappa_m$ and parameters over $\bigcup_{n<m} p_m$. We can ensure at the same time that $\Sigma_\kappa(\mu, i) \cap \kappa_m$ is countably infinite, as is $\Sigma_P(\mu, i) \cap P_m$. (We appeal to this construction in the proof of Lemma 11.)

To construct κ_j we assume an operator $E(\kappa^*, m, j)$ where κ^* is a subset of κ, m is a modality and j is an integer. $E(\kappa^*, m, j)$ is defined as follows. Let α_i enumerate $\{\mu/\mu\alpha : m \in \Sigma \mid \mu \in \Pi(\kappa^*)\}$ and let n_i enumerate integers with infinite repetition: $E(\kappa^*, m, j) = \{\alpha_i | n_i = j\}$.

We construct κ_j as the union of sets κ_j^n using E. $\kappa_j^0 = \emptyset$. $\kappa_j^{n+1} = \kappa_j^n \cup \bigcup_m \bigcup_{k \le j} E(\kappa_k^n, m, j)$. The partition of P proceeds similarly, by distributing $p : \mu$ with $\mu \in \Pi(\kappa_j)$ among P_i with $j \le i$ fairly. ■

Lemma 5 *Let* $\mathbf{C}$ *be an* X*-compatible first-order* S*-consistency property for* Σ*. Define* $\mathbf{C}'$ *as* $S \in \mathbf{C}'$ *just in case there is* $S' \in \mathbf{C}$ *such that* $S \subseteq S'$*. (Thus,* $\mathbf{C}'$ *contains all subsets of members of* $\mathbf{C}$*.) Then* $\mathbf{C}'$ *is also an* X*-compatible first-order consistency property for* Σ*, extending* $\mathbf{C}$ *and closed under subsets.*

Proof. $\mathbf{C}'$ extends $\mathbf{C}$ since for any $S \in \mathbf{C}$, $S \subseteq S$ and hence $S \in \mathbf{C}'$. $\mathbf{C}'$ is closed under subsets because given $S \in \mathbf{C}'$ and $T \subseteq S$, there is an $S' \in \mathbf{C}$ with $S \subseteq S'$; then $T \subseteq S'$ so $T \in \mathbf{C}'$. Now, take $S \in \mathbf{C}'$ and $A \in X$. There is $S' \in \mathbf{C}$ with $S \subseteq S'$; since $\mathbf{C}$ is X-consistent $S' \cup \{\mathbf{t}A^\varepsilon\} \in \mathbf{C}$; since $S \cup \{\mathbf{t}A^\varepsilon\} \subseteq S' \cup \{\mathbf{t}A^\varepsilon\}$, $S \cup \{\mathbf{t}A^\varepsilon\} \in \mathbf{C}'$. Thus $\mathbf{C}'$ is X-compatible.

Finally, the various conditions for being a first-order consistency property for Σ must be checked for $\mathbf{C}'$. $\mathbf{C}'$ introduces no prefixes or first order parameters. so clause 1 is satisfied. To show clause 2, suppose $\mathbf{t}A^\mu$ and $\mathbf{f}A^\mu \in S \in \mathbf{C}'$. Then $\mathbf{t}A^\mu$ and $\mathbf{f}A^\mu \in S' \in \mathbf{C}$, which is impossible.

The reasoning is essentially the same in all the remaining cases: we have $S \in \mathbf{C}'$ and need to show $S \cup T \in \mathbf{C}'$ (for appropriate T). This follows since $S \subseteq S'$ with $S' \in \mathbf{C}$, so $S' \cup T \in \mathbf{C}$ since $\mathbf{C}$ is a consistency property, and $S \cup T \subseteq S' \cup T$. To be concrete, here are representative cases: for $\mathbf{t}A \wedge B^\mu \in S$, we use this argument with $T = \{\mathbf{t}A^\mu, \mathbf{t}B^\mu\}$. For $\mathbf{t}A \vee B^\mu \in S$, we use this argument with T chosen as whichever of $\{\mathbf{t}A^\mu\}$ and $\{\mathbf{t}B^\mu\}$ gives $S' \cup T \in \mathbf{C}$ (we must have one). For $\mathbf{t}\neg A^\mu \in S$, we use this argument with $T = \{\mathbf{f}A^\mu\}$. For $\mathbf{t}\forall x A^\mu \in S$, we have $T = \{\mathbf{t}A[p/x]^\mu\}$ (for any such p where the side conditions apply in in S, they apply in S' because p must occur in S' if p occurs in S and $S \subseteq S'$). We argue similarly for modal universals, such as $\mathbf{t}\Box_i A^\mu$: we have $T = \{\mathbf{t}A^\nu\}$ (any ν that meets the side conditions in S meets them in S' because $S \subseteq S'$). For $\mathbf{t}\exists x A^\mu$, we can pick $T = \{\mathbf{t}A[p/x]^\mu\}$ to find $S' \cup T \in \mathbf{C}$. We argue similarly for modal existence conditions, such as $\mathbf{t}\Diamond_i A^\mu$: we can pick $T = \{\mathbf{t}A^{\mu\alpha}\}$ to find $S' \cup T \in \mathbf{C}$. ■

The simple existential way of dealing with $\exists x$ and $\Diamond_i$ in consistency properties is insufficient for the rest of the completeness proof. It is convenient to reformulate it using a *new parameter condition*.

Definition 23 (Alternate consistency property) *A collection* $\mathbf{C}$ *(of sets of signed prefixed formulas) meets the* new parameter condition *for* Σ *if the following three conditions are met for each* $S \in \mathbf{C}$*.*

- *For each* $\mathbf{t}\exists x A^\mu \in S$*,* $S \cup \{\mathbf{t}A[c/x]^\mu\} \in \mathbf{C}$ *for every first-order parameter* c *that does not occur in* S *and for which* $c : \mu \in \Sigma$*. Likewise, for each* $\mathbf{f}\forall x A^\mu \in S$*,* $S \cup \{\mathbf{f}A[c/x]^\mu\} \in \mathbf{C}$ *for every parameter* c *that does not occur in* S *and for which* $\Sigma \triangleright c : \mu$*.*

- *Whenever we have either (1)* $\mathbf{t}\Diamond_i A^\mu \in S$ *or (2)* $\mathbf{t}\Box_j A^\mu$ *with* $\mathcal{A}(i)$ *KD, KDB, KD4, KD5 or KD45 and with* $i \leq j$ *by* $\mathcal{N}$ *, then* $S \cup \{\mathbf{t}A^{\mu\alpha}\} \in \mathbf{C}$ *for every transition parameter* α *such that:* $\mu/\mu\alpha : i \in \Sigma$*,* α *does not occur in S, and further no parameter* p *occurs in S with* $p : \nu \in \Sigma$ *where* α *occurs in* ν*. Likewise, whenever we have either (1)* $\mathbf{f}\Box_i A^\mu \in S$ *or (2)* $\mathbf{f}\Diamond_j A^\mu \in S$ *with* $\mathcal{A}(i)$ *KD, KDB, KD4, KD5 or KD45 and with* $i \leq j$ *by* $\mathcal{N}$ *, then* $S \cup \{\mathbf{f}A^{\mu\alpha}\} \in \mathbf{C}$ *for every transition parameter* α *such that:* $\mu/\mu\alpha : i \in \Sigma$*,* α *does not occur in S, and further no parameter* p *occurs in S with* $p : \nu \in \Sigma$ *where* α *occurs in* ν*.*

- *If* $\mathbf{u}A^\nu \in S$ *with* $S, \Sigma \triangleright \mu/\nu : j$ *for* $\mathcal{A}(j)$ *one of K5, K45, KD5, KD45 or S5, for* $i \leq j$ *by* $\mathcal{N}$ *and* $\mathcal{A}(i)$ *one of KD, KDB, KD4, KD5 or KD45, then* $S \cup \{\mathbf{t}\top^{\mu\alpha}\} \in \mathbf{C}$ *for every transition parameter* α *which does not occur in S and for which* $\mu/\mu\alpha : i \in \Sigma$*.*

If $\mathbf{C}$ *satisfies all the conditions for being a first-order* $\mathcal{S}$*-consistency property for* Σ*, except for conditions 7, 9 and 10 (on existential and possible statements), and* $\mathbf{C}$ *also satisfies the new parameter condition for* Σ*, we will call* $\mathbf{C}$ *an* alternate $\mathcal{S}$-consistency property for Σ.

To show that this reformulation is general, we consider parameter substitutions.

Definition 24 σ *is a* $\mathcal{S}$-parameter substitution for Σ *(over the language* $L(CONST \cup P)^{\Pi(\kappa)}$*) if* $\sigma = \langle \sigma_P, \sigma_\Pi \rangle$ *where* $\sigma_P : P \to CONST \cup P$ *and* $\sigma_\Pi : \Pi(\kappa) \to \Pi(\kappa)$ *satisfying the following properties:*

- $\sigma_P(p)$ *occurs in* $\Sigma \Rightarrow p$ *occurs in* Σ*;* $\sigma_\Pi(\mu)$ *occurs in* $\Sigma \Rightarrow \mu$ *occurs in* Σ*;*

- *for all c and* μ*, if* $S, \Sigma \triangleright c : \mu$*, then* $S, \Sigma \triangleright \sigma_P(c) : \sigma_\Pi(\mu)$*;*

- *for all* μ *and* ν*, if* $S, \Sigma \triangleright \mu/\nu : i$*, then* $S, \Sigma \triangleright \sigma_\Pi(\mu)/\sigma_\Pi(\nu) : i$*.*

For formula A we can write $\sigma_P(A)$ *for the formula obtained by replacing each occurrence of first-order parameter in A by an occurrence of its image under* σ_P*. Then we can write* $\sigma(A^\mu)$ *for* $\sigma_P(A)^{\sigma_\Pi(\mu)}$*; we extend* σ *to sets of signed prefixed formulas accordingly.*

Note that a parameter substitution σ, unlike a syntactic substitution of values to variables, may have infinitely many symbols p for which $\sigma(p)$ differs from p.

Lemma 6 *Let* Σ *be a fair typing (with respect to* κ *and P). Suppose* $\mathbf{C}'$ *is an X-compatible first-order* $\mathcal{S}$*-consistency property for* Σ*, with* $\mathbf{C}'$ *closed under subsets. Define* $\mathbf{C}''$ *by* $S \in \mathbf{C}''$ *just in case* $\sigma(S) \in \mathbf{C}'$ *for some parameter substitution* σ *for* Σ*. Then* $\mathbf{C}''$ *is an X-compatible alternate* $\mathcal{S}$*-consistency property for* Σ *that extends* $\mathbf{C}'$ *and is closed under subsets.*

Proof. First, we show that $\mathbf{C}''$ is an alternate S-consistency property, condition by condition. To show clause 1, suppose we have $S \in \mathbf{C}''$ in virtue of σ; by consistency of $\mathbf{C}'$ any prefix $\sigma_\Pi(\mu)$ or first order parameter $\sigma_P(p)$ that occurs in $\sigma(S)$ occurs in Σ; then by definition of parameter substitution μ or p occurs in Σ. To show clause 2, suppose $\mathbf{t}A^\mu \in S$ and $\mathbf{f}A^\mu \in S$ for $S \in \mathbf{C}''$. Then $\mathbf{t}\sigma(A^\mu) \in \sigma(S)$ and $\mathbf{f}\sigma(A^\mu) \in \sigma(S)$ for $\sigma(S) \in \mathbf{C}'$—impossible since $\mathbf{C}'$ is a first-order S-consistency property.

To show the remaining relevant clauses from Definition 21, we again use essentially common reasoning to show, given $S \in \mathbf{C}''$, that $S \cup T \in \mathbf{C}''$ for appropriate T. We get this by showing that $\sigma(S \cup T) = \sigma(S) \cup \sigma(T) \in \mathbf{C}'$; in each case this argument rests on the observations that $\sigma(S) \in \mathbf{C}'$ and $\sigma(S)$ triggers the relevant clause of the S-consistency set definition.

For example, given $\mathbf{t}A \wedge B^\mu \in S \in \mathbf{C}''$, we need $S \cup \{\mathbf{t}A^\mu, \mathbf{t}B^\mu\} \in \mathbf{C}''$. This is established by the definition of $\mathbf{C}''$ as follows: Since $\sigma(\mathbf{t}A \wedge B^\mu) = \mathbf{t}\sigma_P(A) \wedge \sigma_P(B)^{\sigma_\Pi(\mu)}$ and $\sigma(S)$ is an element of consistency property $\mathbf{C}'$, we obtain $\sigma(S) \cup \{\sigma(\mathbf{t}A^\mu), \sigma(\mathbf{t}B^\mu)\} \in \mathbf{C}'$. Similar reasoning applies for $\mathbf{t}\neg A^\mu \in S \in \mathbf{C}''$. Next, given $\mathbf{t}A \vee B^\mu \in S \in \mathbf{C}''$, we need $S \cup \{\mathbf{t}A^\mu\} \in \mathbf{C}''$ or $S \cup \{\mathbf{t}B^\mu\} \in \mathbf{C}''$. But $\sigma(S) \in \mathbf{C}''$ and $\mathbf{t}\sigma_P(A)^{\sigma_\Pi(\mu)} \vee \sigma_P(B)^{\sigma_\Pi(\mu)} \in \sigma(S)$ so either $\sigma(S) \cup \sigma(\mathbf{t}A^\mu) \in \mathbf{C}'$ or $\sigma(S) \cup \sigma(\mathbf{t}B^\mu) \in \mathbf{C}'$.

Next, for the universal conditions, suppose $\mathbf{t}\forall x A^\mu \in S \in \mathbf{C}''$. We need to show $S \cup \{\mathbf{t}A[p/x]^\mu\} \in \mathbf{C}''$ for p a constant or $p \in P$ such that p occurs in S and $S, \Sigma \triangleright p : \mu$. We know since σ is a parameter substitution that either $\sigma_P(p)$ is a constant or $\sigma_P(p)$ occurs in $\sigma(S)$ and $S, \Sigma \triangleright \sigma_P(p) : \sigma_\Pi(\mu)$. Thus from $\sigma(\mathbf{t}\forall x A^\mu) \in \sigma(S)$ we get $\sigma(S) \cup \{\mathbf{t}\sigma_P(A[p/x])^{\sigma_\Pi(\mu)}\} \in \mathbf{C}'$. Finally, for $\mathbf{t}\Box_i A^\mu \in S \in \mathbf{C}''$, we consider ν that occurs in S with $S, \Sigma \triangleright \mu/\nu : i$. Then $\sigma_\Pi(\nu)$ occurs in $\sigma(S)$ and (since σ is a parameter substitution) $S, \Sigma \triangleright \sigma_\Pi(\mu)/\sigma_\Pi(\nu) : i$. Hence $\sigma(S) \cup \{\mathbf{t}\sigma_P(A)^{\sigma_\Pi(\nu)}\} \in \mathbf{C}'$.

Now consider the new parameter condition, first clause; for reference we describe the positive condition. We suppose $\mathbf{t}\exists x A^\mu \in S \in \mathbf{C}''$. Then $\mathbf{t}\sigma(\exists x A^\mu) \in \sigma(S) \in \mathbf{C}'$ and for some t such that $S, \Sigma \triangleright t : \sigma(\mu)$, $\sigma(S) \cup \{\mathbf{t}\sigma(A^\mu)\} \in \mathbf{C}'$. Now let p be any first-order parameter that does not occur in S with $p : \mu \in \Sigma$, and define $\sigma' = \langle \sigma'_P, \sigma_\Pi \rangle$ with σ'_P exactly like σ_P except possibly that $\sigma'_P(p) = t$. We claim σ' is a parameter substitution. Since $\sigma'_P(p) = t$ is the only new assignment, we need only show that $S, \Sigma \triangleright p : \nu \Rightarrow S, \Sigma \triangleright \sigma'_P(p) : \sigma_\Pi(\nu)$. We can show this by induction on the height of the typing derivation for $S, \Sigma \triangleright p : \nu$. The base case has $p : \nu \in \Sigma$, but since Σ is fair and $p : \mu \in \Sigma$ already, we must have $\nu = \mu$ and by construction $S, \Sigma \triangleright \sigma'_P(p) : \sigma_\Pi(\mu)$. The result extends straightforwardly to compound derivations using the fact that σ' agrees with σ on prefixes and the fact that σ is a parameter substitution. Since p does not occur in S, $\sigma'(S) = \sigma(S)$. But now we have $\sigma'(S \cup \{\mathbf{t}A[p/x]^\mu\}) \in \mathbf{C}'$, so we have established $S \cup \{\mathbf{t}A[p/x]^\mu\} \in \mathbf{C}''$ as needed.

The reasoning for the second and third clauses extends this reasoning. For instance, for $\mathbf{t}\Diamond_i A^\mu \in S \in \mathbf{C}''$ we need to show $S \cup \mathbf{t}A^{\mu\alpha}$ for any parameter α where $\mu/\mu\alpha : i \in \Sigma$ and S has occurrences neither of α nor of any p with $p : \nu \in \Sigma$ and α occurs in ν.

Now, because $\mathbf{C}'$ is a consistency property, we know $\sigma(S) \cup \{\mathbf{t}\sigma(A)^{\sigma_\Pi(\mu)\beta}\} \in \mathbf{C}'$ for some β such that $\sigma_\Pi(\mu)/\sigma_\Pi(\mu)\beta : i \in \Sigma$. We will extend σ_Π by establishing a correspondence between prefixes involving α and prefixes containing β. In particular, define

$$\Pi_O = \{\mu\alpha\nu \in \Pi(\kappa) \mid \text{ for some } \nu\}$$
$$\Pi_N = \{\sigma_\Pi(\mu)\beta\nu \in \Pi(\kappa) \mid \text{ for some } \nu\}$$

Since Σ is fair, we can construct a function $\rho : \Pi_O \to \Pi_N$ such that $\rho(\mu\alpha) = \sigma(\mu)\beta$ and such that for any $\xi, \zeta \in \Pi_O$, if $\xi/\zeta : i \in \Sigma$, $\rho(\xi)/\rho(\zeta) : i \in \Sigma$. Since α does not occur in S, no element of Π_O occurs in S. We can define σ'_Π by

$$\sigma'_\Pi(\xi) = \begin{cases} \rho(\xi) \text{ if } \xi \in \Pi_O \\ \sigma_\Pi(\xi) \text{ otherwise} \end{cases}$$

And we observe that σ'_Π agrees with σ_Π on all prefixes that occur in S. To obtain a parameter substitution, we will also adjust $\sigma_P(p)$ for all p with $p : \xi \in \Sigma$ for $\xi \in \Pi_O$. Since Σ is fair, both $\Sigma_P(\xi)$ and $\Sigma_P(\rho(\xi))$ are countable sets: let $g_\xi : \Sigma_P(\xi) \to \Sigma_P(\rho(\xi))$ be any one-to-one onto map. Then define σ'_P by

$$\sigma'_P(p) = \begin{cases} g_\xi(p) \text{ if } p : \xi \in \Sigma \text{ and } \xi \in \Pi_O \\ \sigma_P(p) \text{ otherwise} \end{cases}$$

Again, we can observe that since S contains occurrences of no p such that $p : \nu \in \Sigma$ with α in ν, σ'_P agrees with σ_P on all parameters that occur in S.

We claim that $\sigma' = \langle \sigma'_P, \sigma'_\Pi \rangle$ so defined is a parameter substitution. First we show, by induction, $S, \Sigma \triangleright \mu'/\nu' : j \Rightarrow S, \Sigma \triangleright \sigma'_\Pi(\mu')/\sigma'_\Pi(\nu') : j$.

The base case considers a tuple $\xi/\zeta : i \in \Sigma$. Suppose $\zeta \in \Pi_O$. If $\zeta = \mu\alpha$ then $\xi = \mu$ and $\sigma'_\Pi(\mu) = \sigma(\mu)$ while $\sigma(\zeta) = \sigma(\mu)\beta$ and $\sigma(\mu)/\sigma(\mu)\beta : i \in \Sigma$ by assumption. Otherwise $\xi = \mu\alpha\nu$; $\sigma'_\Pi(\xi) = \rho(xi)$ and $\rho(\xi) : \rho(\zeta) : i$ by construction. Otherwise $\zeta \notin \Pi_O$. Neither is $\xi \in \Pi_O$ so $\sigma'_\Pi(\xi) = \sigma(\xi)$ and $\sigma'_\Pi(\zeta) = \sigma(\zeta)$ so since σ is a parameter substitution $S, \Sigma \triangleright \sigma(\xi)/\sigma(\zeta) : i$. The inductive cases follow straightforwardly (as in the proof of Lemma 2).

Next we show by induction that $S, \Sigma \triangleright c : \mu' \Rightarrow S, \Sigma \triangleright \sigma'_P(c) : \sigma'_\Pi(\mu')$. Here the argument mirrors the reasoning for the existential case. The base case has $c : \xi \in \Sigma$. Then if $\xi \in \Pi_O$ then $g_\xi(c) : \rho(\xi) \in \Sigma$ by construction; otherwise this is a case where σ' agrees with parameter substitution σ. The result extends to compound derivations using the fact that σ'_Π sends related prefixes to related prefixes.

Moreover, $\sigma'(S \cup \{\mathbf{t}A^{\mu\alpha}\}) = \sigma(S) \cup \{\mathbf{t}\sigma(A)^{\sigma(\mu)\beta}\}$, since σ'_Π is identical to σ_Π on prefixes in S, and σ'_P is identical to σ_P on first-order parameters in S.

Having established that $\mathbf{C}''$ is an alternate S-consistency condition, we turn to the remaining facts. $\mathbf{C}''$ extends $\mathbf{C}'$ since the pair $\langle 1_P, 1_\Pi \rangle$ consisting of the identity map on first-order parameters and the identity on prefixes is a S-parameter substitution. $\mathbf{C}''$ is X-compatible because $\mathbf{C}'$ is X-compatible and moreover X is a set of sentences from $L(\mathrm{CONST})$ and hence $\sigma(X) = X$ for any S-parameter substitution. $\mathbf{C}''$ is closed under subsets because $\mathbf{C}'$ is closed under subsets and $S \subseteq S'$ implies $\sigma(S) \subseteq \sigma(S')$. ■

A collection $\mathbf{C}$ of sets is said to be of *finite character* provided $S \in \mathbf{C}$ if and only if every finite subset of S belongs to $\mathbf{C}$.

Lemma 7 *Suppose $\mathbf{C}''$ is an alternate S-consistency property that is X-compatible and closed under subsets. Let $\mathbf{C}'''$ consist of those sets S all of whose finite subsets are in $\mathbf{C}''$. Then $\mathbf{C}'''$ is again an X-compatible alternate S-consistency property that extends $\mathbf{C}''$ and is of finite character.*

Proof. First, we show that $\mathbf{C}''$ is an alternate S-consistency property, condition by condition. To show clause 1, suppose $\mathbf{t}A^{\mu} \in S$ and $\mathbf{f}A^{\mu} \in S$ for $S \in \mathbf{C}'''$. Then $S_0 = \{\mathbf{t}A^{\mu}, \mathbf{f}A^{\mu}\}$ is a finite subset of S: therefore $S_0 \in \mathbf{C}''$, which is impossible.

For the remaining conditions, for $S \in \mathbf{C}'''$ we need to show $S \cup T \in \mathbf{C}'''$ (for appropriate T). We derive a contradiction from the assumption of some finite $F \subset S \cup T$ with $F \notin \mathbf{C}''$ by constructing finite $H \subset S$ for which $H \cup T \in \mathbf{C}''$ and $F \subseteq H \cup T$. (Such argument also shows that $\mathbf{C}'''$ is X-compatible.)

For example, suppose $\mathbf{t}A \wedge B^{\mu} \in S$. For $T = \{\mathbf{t}A^{\mu}, \mathbf{t}B^{\mu}\}$ we must show $S \cup T \in \mathbf{C}'''$. Suppose otherwise: then there is a finite $F \subseteq S \cup T$ with $F \notin \mathbf{C}''$. But consider $H = (F \cap S) \cup \{\mathbf{t}A \wedge B^{\mu}\}$. $H \subseteq S$ and H is finite, so $H \in \mathbf{C}''$, so $H \cup T \in \mathbf{C}''$. $\mathbf{C}''$ is closed under subsets, and $F \subseteq (F \cap S) \cup T$, so $F \in \mathbf{C}''$. This is a contradiction. (The same goes for $\mathbf{t}\neg A^{\mu} \in S$.)

For $\mathbf{t}\forall x A^{\mu} \in S$, we consider p constant or p occurs in S with $S, \Sigma \triangleright p : \mu$. If p occurs in S it occurs in some particular expression $E \in S$. We want to show $S \cup T \in \mathbf{C}'''$ for $T = \{\mathbf{t}A[p/x]^{\mu}\}$. Suppose otherwise: then there is $F \subseteq S \cup T$ with $F \notin \mathbf{C}''$. We can now use $H = (F \cap S) \cup \{\mathbf{t}\forall x A^{\mu}, E\}$ to show $F \subseteq H \cup T \in \mathbf{C}''$, a contradiction. For $\mathbf{t}\exists x A^{\mu}$, we apply this reasoning with $T = \{\mathbf{t}A[p/x]^{\mu}\}$ for $S, \Sigma \triangleright p : \mu$ and p not occurring in S; we hypoethesize $H = (F \cap S) \cup \{\mathbf{t}\exists x A^{\mu}\}$ (where surely p does not occur). These two schemas also extend to the various $\Box_i$ and $\Diamond_i$ cases.

Finally, for $\mathbf{t}A \vee B^{\mu} \in S$, let $T_1 = \{\mathbf{t}A^{\mu}\}$ and $T_2 = \{\mathbf{t}B^{\mu}\}$. Assuming *neither* $S \cup T_1 \in \mathbf{C}'''$ nor $S \cup T_2 \in \mathbf{C}'''$ gives $F_1 \subseteq S \cup T_1$ and $F_2 \subseteq S \cup T_2$: we take H as $(F_1 \cap S) \cup (F_2 \cap S)$.

We now establish the remaining claims about $\mathbf{C}'''$. The fact that $\mathbf{C}''$ is closed under subsets ensures that $S \in \mathbf{C}''$ implies $S \in \mathbf{C}'''$. $\mathbf{C}'''$ is of finite character because $\mathbf{C}''$ and $\mathbf{C}'''$ agree on finite sets. ■

Lemmas 5, 6 and 7 are summarized in Proposition 8.

Proposition 8 *Let Σ be a fair typing and let $\mathbf{C}$ be an X-compatible first-order S-consistency property for Σ (where X is a set of sentences of $L(CONST)$). Then $\mathbf{C}$ may be extended to a collection $\mathbf{C}^*$ that is an X-compatible alternate S-consistency property for Σ of finite character.*

It will be convenient to construct certain additional alternate S-consistency properties from $\mathbf{C}^*$.

Definition 25 (Sections) *Let P' be a set of first-order parameters and let κ' be a set of modal parameters. The $\langle P', \kappa' \rangle$-*section *of a collection $\mathbf{C}^*$ is a collection $C^*|^{P',\kappa'}$ defined as $\{S \in \mathbf{C}^* \mid$ all members of S are signed expressions over $L(CONST \cup P')^{\Pi(\kappa')}\}$.*

Lemma 9 (Sections) *If $\mathbf{C}^*$ is an X-compatible alternate S-consistency property of finite character for $L(CONST \cup P)^{\Pi(\kappa)}$, then any $\mathbf{C}^*|^{P',\kappa'}$ is an X-compatible alternate S-consistency property for $L(CONST \cup P')^{\Pi(\kappa')}$—so long as $P' \subseteq P$ and $\kappa' \subseteq \kappa$.*

Proof. The argument here is straightforward. For any $S \in \mathbf{C}^*|^{P',\kappa'}$, $S \in \mathbf{C}^*$. So there can be no $\mathbf{t}A^\mu \in S$ with $\mathbf{f}A^\mu \in S$. The remaining clauses of the alternate S-consistency property (and X-compatibility) require $S \cup T \in \mathbf{C}^*|^{P',\kappa'}$ for some appropriate T a signed expression of $L(\mathrm{CONST} \cup P')^{\Pi(\kappa')}$; but we already have $S \cup T \in \mathbf{C}^*$. Finally, since $\mathbf{C}^*$ is of finite character: $S \in \mathbf{C}^*$ and S a set of signed expression of $L(\mathrm{CONST} \cup P')^{\Pi(\kappa')}$ just in case every finite subset F of S has $F \in \mathbf{C}^*$ and F a signed expression of $L(\mathrm{CONST} \cup P')^{\Pi(\kappa')}$. But this is equivalent to $S \in \mathbf{C}^*|^{P',\kappa'}$ just in case every finite subset of S belongs to $\mathbf{C}^*|^{P',\kappa'}$. ■

2.5.2 Model Existence

In this subsection, we show that our construction of S-consistency properties gives us—for any S-consistency property and any set that belongs to the consistency property—a model in which the set is satisfied. We first establish two facts and a definition that we will use in the construction.

Proposition 10 *In an alternate S-consistency property $\mathbf{C}^*$ of finite character: the union of any chain of members is again a member; for any $S \in \mathbf{C}^*$ there is a maximal $S' \in \mathbf{C}^*$ with $S \subseteq S'$.*

Proof. Let $S_0 \subseteq S_1 \ldots$ be a chain of members of $\mathbf{C}^*$, and let S be its union. We want to show $S \in \mathbf{C}^*$. Suppose not; since $\mathbf{C}^*$ is of finite character, there must be some finite $F \subseteq S$ with $F \notin \mathbf{C}^*$. But since F is finite there is some element S_n of the chain such that $F \subseteq S_n$. This contradicts the assumption that S_n is a member of $\mathbf{C}^*$.

We now have that the union of a chain of members of $\mathbf{C}^*$ extending S is also a member of $\mathbf{C}^*$ that extends S. Thus, we can apply Zorn's Lemma to the set $\{S' \in \mathbf{C}^* \mid S \subseteq S'\}$ to obtain the needed maximal element. ■

Definition 26 (Saturation) *Let Σ be fair, and suppose $\mathbf{C}^*$ is an X-compatible alternate S-consistency property for Σ (over $L(CONST \cup P)^{\Pi(\kappa)}$) of finite character. Let S and T be sets of signed expressions of $L(CONST \cup P')^{\Pi(\kappa')}$ (with $P' \subseteq P$ and $\kappa' \subseteq \kappa$). We say S is* downward S-saturated into T *in $\mathbf{C}^*|^{P',\kappa'}$ just in case the following conditions are met:*

1. *If $\mathbf{t}\exists x A^\mu \in S$, then $\mathbf{t}A[c/x]^\mu \in T$ for some $c \in P'$ with $S, \Sigma \triangleright c : \mu$; likewise if $\mathbf{f}\forall x A^\mu \in S$, then $\mathbf{f}A[c/x]^\mu \in T$ for some $c \in P'$ with $S, \Sigma \triangleright c : \mu$.*

2. *If either* $\mathbf{t}\Diamond_i A^\mu \in S$ *or* $\mathbf{t}\Box_j A^\mu \in S$ *with* $A(i)$ *KD, KDB, KD4, KD5 or KD45 and with* $i \leq j$ *by* N *, then* $\mathbf{t}A^{\mu\alpha} \in T$ *for some* $\alpha \in \kappa'$ *with* $S,\Sigma \triangleright \mu/\mu\alpha : i$*; likewise if either* $\mathbf{f}\Box_i A^\mu \in S$ *or* $\mathbf{f}\Diamond_j A^\mu \in S$ *with* $A(i)$ *KD, KDB, KD4, KD5 or KD45 and with* $i \leq j$ *by* N *, then* $\mathbf{f}A^{\mu\alpha} \in T$ *for some* $\alpha \in \kappa'$ *with* $S,\Sigma \triangleright \mu/\mu\alpha : i$.

3. *If* $\mathbf{u}A^\nu \in S$ *with* $S,\Sigma \triangleright \mu/\nu : j$ *for* $A(j)$ *one of K5, K45, KD5, KD45 or S5, for* $i \leq j$ *by* N *and* $A(i)$ *one of KD, KDB, KD4, KD5 or KD45, then* $\mathbf{t}\top^{\mu\alpha} \in T$ *for some parameter* $\alpha \in \kappa$ *with* $S,\Sigma \triangleright \mu/\mu\alpha : i$.

S is downward S-saturated *in* $\mathbf{C}^*|^{P',\kappa'}$ *just in case S is maximal in* $\mathbf{C}^*|^{P',\kappa'}$ *and S is downward* S*-saturated into S.*

Lemma 11 *Let* $\mathbf{C}^*$ *be an X-compatible alternate* S*-consistency property of finite character for* Σ *(over* $L(CONST \cup P)^{\Pi(\kappa)}$*); let Q and R be two disjoint countably infinite subsets of P, and let* ξ *and* ζ *be two disjoint countably infinite subsets of* κ*, such that* Σ *is fair to R and* ζ*. If* $S \in \mathbf{C}^*|^{Q,\xi}$ *then S may be extended to a set that is downward* S*-saturated in* $\mathbf{C}^*|^{Q\cup R,\xi\cup\zeta}$.

Proof. We begin by defining an operator F. The input of F is a set of signed prefixed formulas $T \in \mathbf{C}^*|^{P',\kappa'}$ for some language $L(\mathrm{CONST} \cup P')^{\Pi(\kappa')}$ and countable sets P'' of first-order parameters and κ'' of modal parameters disjoint from P' and κ' subject to two conditions against the typing Σ. First, Σ is fair to $P' \cup P''$ and $\kappa' \cup \kappa''$. Second, $\Sigma_\kappa(\mu, i) \cap \kappa''$ is countably infinite, as is $\Sigma_P(\mu, i) \cap P''$. The output of F, $F(T, P'', \kappa'')$ is a set T' that is maximal in $\mathbf{C}^*|^{P'\cup P'',\kappa'\cup\kappa''}$, where T is downward S-saturated into T' in $\mathbf{C}^*|^{P',\kappa'}$.

In brief, F adds to T a witness for each existential and possible signed prefixed formula in T, and extends the result to a maximal set. Let $T_\exists$ be the signed prefixed formulas in T of the form $\mathbf{t}\exists x A^\mu$ or $\mathbf{f}\forall x A^\mu$. Enumerate $T_\exists$, and assign all expressions $E \in T_\exists$ a unique first-order parameter $p_E \in P''$ with $p_E : \mu \in \Sigma$. This is possible because Σ meets the two conditions provided and $T_\exists$ is countable. Now we define $S_\exists$ to be $\{\mathbf{t}A[p_E/x]^\mu \mid E = \mathbf{t}\exists x A^\mu \in T_\exists\} \cup \{\mathbf{f}A[p_E/x]^\mu \mid E = \mathbf{f}\forall x A^\mu \in T_\exists\}$. Our enumeration of $T_\exists$ induces an enumeration of $S_\exists$; let $S_{n\exists}$ denote the first n elements of $S_\exists$ in this enumeration.

Similarly, let $T_\Diamond$ be the signed prefixed formulas in T of one of the five following forms: $\mathbf{t}\Diamond_i x A^\mu$; $\mathbf{t}\Box_j x A^\mu$ for $A(i)$ KD, KDB, KD4, KD5 or KD45 with $i \leq j$ by N; $\mathbf{f}\Box_i A^\mu$; $\mathbf{f}\Diamond_j A^\mu$ for $A(i)$ KD, KDB, KD4, KD5 or KD45 with $i \leq j$ by N; and $\mathbf{u}A^\nu$ with $S,\Sigma \triangleright \mu/\nu : j$ for $A(j)$ one of K5, K45, KD5, KD45 or S5, for $i \leq j$ by N and $A(i)$ one of KD, KDB, KD4, KD5 or KD45. Enumerate $T_\Diamond$, and assign all expressions $E \in T_\Diamond$ a unique modal parameter $\alpha_E \in \kappa''$ with $\mu/\mu\alpha : i \in \Sigma$. Again, this is possible because Σ meets the two conditions provided and $T_\Diamond$ is countable. Now we define $S_\Diamond$ to be $\{\mathbf{t}A^{\mu\alpha_E} \mid E = \mathbf{t}\Diamond_i A^\mu \in T_\Diamond$ or $E = \mathbf{t}\Box_j A^\mu \in T_\Diamond\} \cup \{\mathbf{f}A^{\mu\alpha_E} \mid E = \mathbf{f}\Box_i A^\mu \in T_\Diamond$ or $E = \mathbf{f}\Diamond_j A^\mu \in T_\Diamond\} \cup \{\mathbf{t}\top^{\mu\alpha_E} \mid E = \mathbf{u}A^\nu \in T_\Diamond$ with $S,\Sigma \triangleright \mu/\nu : j\}$. Our enumeration of $T_\Diamond$ induces an enumeration of $S_\Diamond$; let $S_{n\Diamond}$ denote the first n elements of $S_\Diamond$ in this enumeration.

We claim that $T \cup S_\exists \cup S_\diamond \in \mathbf{C}^*|^{P' \cup P'', \kappa' \cup \kappa''}$. We know $T \in \mathbf{C}^*|^{P' \cup P'', \kappa' \cup \kappa''}$, so if the claim is false there must be a first n such that $T \cup S_{n+1\exists} \cup S_{n+1\diamond} \notin \mathbf{C}^*|^{P' \cup P'', \kappa' \cup \kappa''}$. But we can get from $T \cup S_{n\exists} \cup S_{n\diamond}$ to $T \cup S_{n+1\exists} \cup S_{n+1\diamond}$ by two steps of applying the new parameter condition for the alternate consistency property $\mathbf{C}^*|^{P' \cup P'', \kappa' \cup \kappa''}$. This is a contradiction. As the result of $\mathcal{F}$, take a maximal member T' of $\mathbf{C}^*|^{P' \cup P'', \kappa' \cup \kappa''}$ that extends $T \cup S_\exists \cup S_\diamond$.

As claimed, it follows from this construction that T' is maximal in $\mathbf{C}^*|^{P' \cup P'', \kappa' \cup \kappa''}$ and that T is downward $\mathcal{S}$-saturated into T' in $\mathbf{C}^*|^{P', \kappa'}$.

Now we describe an increasing sequence of sets $S_1, \ldots$; we construct the desired set S^* as $\cup_n S_n$. We partition R and ζ into countably many disjoint countable sets $R_1, \ldots$ and $\zeta_1, \ldots$ according to Σ, using the construction following Definition 22; we define R^*_n as $\cup_{m \le n} R_m$ and ζ^*_n as $\cup_{m \le n} \zeta_m$. S_1 is simply $\mathcal{F}(S, R_1, \zeta_1)$. Given S_n, $S_{n+1} = \mathcal{F}(S_n, R_n, \zeta_n)$. By construction of $\mathcal{F}$, clearly S_n is a maximal set in in $\mathbf{C}^*|^{R^*_n, \zeta^*_n}$ and S_n is downward $\mathcal{S}$-saturated *into* S_{n+1} in $\mathbf{C}^*|^{R^*_n, \zeta^*_n}$.

$S^* \in \mathbf{C}^*|^{Q \cup R, \xi \cup \zeta}$ by Proposition 10, because S^* is the union of a chain in $\mathbf{C}^*|^{Q \cup R, \xi \cup \zeta}$ which is an alternate consistency property of finite character by Lemma 9.

S^* must be maximal in $\mathbf{C}^*|^{Q \cup R, \xi \cup \zeta}$. Since $\mathbf{C}^*|^{Q \cup R, \xi \cup \zeta}$ is of finite character, it suffices to show for any E such that $S^* \cup E \in \mathbf{C}^*|^{Q \cup R, \xi \cup \zeta}$, $E \in S^*$. Consider an E that meets the hypothesis; E is a signed expression of $L(\text{CONST} \cup Q \cup R)^{\Pi(\xi \cup \zeta)}$ and can only contain a finite number of parameters; E is therefore in fact a signed expression in $L(\text{CONST} \cup Q \cup R^*_n)^{\Pi(\xi \cup \zeta^*_n)}$ for some n. It follows that $S_n \cup \{E\} \in \mathbf{C}^*|^{Q \cup R^*_n, \xi \cup \zeta^*_n}$. But since S_n is maximal here by construction, we must have $E \in S_n$ and hence $E \in S^*$.

By analogous reasoning, for any existential or possible signed prefixed formula E in S^*, there is an S_n at which E first appears. There is hence a witness for E in S_{n+1} and thus in S^*. ■

Definition 27 (Term Frame) *Let S be a downward $\mathcal{S}$-saturated set of signed expressions of some language $L(CONST \cup P)^{\Pi(\kappa)}$ for some typing Σ. Define the* term frame *of S as a tuple $\langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ as follows:*

- $\mathcal{G} = \{\mu \mid \mathbf{u}A^\mu \in S\}$.
- $\mathcal{R}_i = \{\langle \mu, \nu \rangle \mid \mu \in \mathcal{G}, \nu \in \mathcal{G}, S, \Sigma \triangleright \mu/\nu : i\} \cup \{\langle \mu, \mu \rangle \mid \mu \in \mathcal{G}, \mathcal{A}(j)$ *is KD, KDB, KD4, KD5 or KD45, $j \le i$ by $\mathcal{N}$ and there is no $\nu \in \mathcal{G}$ such that $S, \Sigma \triangleright \mu/\nu : j\}$.*
- $\mathcal{D}(\mu) =$*CONST*$\cup \{t \mid t$ *occurs in S and* $S, \Sigma \triangleright t : \mu\}$

Lemma 12 (Respect) *Let $\mathcal{F}$ be the term frame for S (a downward $\mathcal{S}$-saturated set) according to Σ. Then $\mathcal{F}$ respects the regime $\mathcal{S}$.*

Proof. We begin with an observation. Suppose there is a prefix μ in S for which no ν occurs in S with $S, \Sigma \triangleright \mu/\nu : i$ with $\mathcal{A}(i)$ KD, KDB, KD4, KD5 or KD45. Then for any j with $i \le j$ and $\mathcal{A}(j)$ in K5, K45, KD5, KD45 or S5, no ν' occurs in S with

$S, \Sigma \triangleright \mu/\nu' : j$. Since S is downward S-saturated, if there was such a ν', by saturation we would have $\mathbf{t}\top^{\mu\alpha} \in S$ for some modal parameter α with $\mu/\mu\alpha : i$. This contradicts our supposition.

Now we can consider the conditions for F to respect S case by case.

- $A(i)$ is T, B, S4 or S5. Since the consistency property ensures that any world μ occurs in Σ, by rule (T) there is a typing derivation $S, \Sigma \triangleright \mu/\mu : i$. Thus, R_i is reflexive.

- $A(i)$ is KB, KDB or B, and $\mu R_i \nu$. There are two cases: in one case there is a derivation $S, \Sigma \triangleright \mu/\nu : i$ so by rule (B) there is a derivation $S, \Sigma \triangleright \nu/\mu : i$: R_i is symmetric; in the other case, we have the obviously symmetric tuple $\nu = \mu$.

- $A(i)$ is K4, K45, KD4, KD45, S4 or S5. Suppose $\mu R_i \mu'$ and $\mu' R_i \nu$. There are three cases: in one case, there are derivations $S, \Sigma \triangleright \mu/\mu' : i$ and $S, \Sigma \triangleright \mu'/\nu : i$, and so by rule (4) there is a derivation $S, \Sigma \triangleright \mu/\nu : i$: R_i is transitive; in the other cases, $\mu = \mu'$ or $\mu' = \nu$ so transitivity holds by assumption.

- $A(i)$ is KD, KDB, KD4, KD5 or KD45. Then R_i is serial, because either $\mu R_i \nu$ for some prefix ν or else by construction $\mu R_i \mu$.

- $A(i)$ is K5, K45, KD5, KD45 or S5. Suppose $\mu R_i \mu'$ and $\mu R_i \nu$. There are three cases. In the first case, there are derivations $S, \Sigma \triangleright \mu/\mu' : i$ and $S, \Sigma \triangleright \mu/\nu : i$, and so by rule (5) there is a derivation $S, \Sigma \triangleright \mu/\nu : i$. In the second case, $\mu' = \mu$ (accessible by construction): then the hypothesis $\mu R_i \nu$ is the conclusion needed for euclideanness. Finally, there is the possibility that $\nu = \mu$ (accessible by construction). But then by our observation, we must have $\mu' = \mu$ as well; the hypothesis $\mu R_i \mu$ is the needed conclusion.

- Suppose $i \leq j$ and $\mu R_i \nu$. There are two cases. If $\nu = \mu$ (accessible by construction) then there must be some modality k with $k \leq i$, $A(k)$ one of KD, KDB, KD5 or KD45, and no $\nu \in G$ such that $S, \Sigma \triangleright \mu/\nu : j$. But since $i \leq j$, we have $k \leq j$ by transitivity of N and hence $\mu R_j \mu$ by construction as well. In the second case, we have a derivation of $S, \Sigma \triangleright \mu/\nu : i$; from this we derive the needed relation by (Inc).

- Suppose Q is *constant*. Then $S, \Sigma \triangleright t : \mu$ for any t that occurs in S, and any $\mu \in G$, thus $D(\mu) = D(\nu)$ for any $\mu, \nu \in G$. And suppose Q is *increasing*. Obviously we need only consider $\mu R_i \nu$ with $\mu \neq \nu$. But in this case we must have $S, \Sigma \triangleright \mu/\nu : i$. Now $t \in D(\mu)$ implies $S, \Sigma \triangleright t : \mu$; putting the two derivations together by (I) gives $t \in D(\nu)$.

This concludes the proof. ■

Definition 28 (Term Model) *Let S be a downward S-saturated set of signed expressions of some language $L(CONST \cup P)^{\Pi(\kappa)}$ for some typing Σ. Let $\langle G, R, D \rangle$ be the term frame of S, and define an interpretation J as follows:*

1. *For each constant $c \in CONST$, $J(c) = c$;*
2. *For any $t_1, \ldots, t_n \in CONST \cup P$, $\langle t_1, \ldots, t_n \rangle \in J(R_i)$ at world μ if and only if $\mathbf{t}R_i(t_1, \ldots, t_n)^{\mu} \in S$.*

Then $M = \langle G, R, D, J \rangle$ is a k-modal model, called the term model *for S.*

Lemma 13 (Satisfaction) *Let $M = \langle G, R, D, J \rangle$ be the term model for S downward saturated in $\mathbf{C}^*$. Let g be an assignment that is the identity on parameters. Then for each signed prefixed formula $\mathbf{t}A^{\mu} \in S$, $M, \mu \Vdash_g A$; conversely, for each signed prefixed formula $\mathbf{f}A^{\mu} \in S$, $M, \mu \nVdash_g A$.*

Proof. The proof is by induction on the degree of the formula A. In the base case, we have $\mathbf{u}A^{\mu}$ with A atomic formula; there are in principle four possibilities.

- $\mathbf{u} = \mathbf{t}$, $A = \top$: Then always $M, \mu \Vdash_g A$.
- $\mathbf{u} = \mathbf{t}$, $A = R_i(t_1, \ldots, t_n)$: Then $M, \mu \Vdash_g A$ if and only if $t_1, \ldots, t_n \in J(R_i)$ at μ (since g is the identity on parameters and J is the identity on constants). This holds by construction.
- $\mathbf{u} = \mathbf{f}$, $A = \top$: This is impossible, by the definition of a consistency property.
- $\mathbf{u} = \mathbf{f}$, $A = R_i(t_1, \ldots, t_n)$: Suppose for contradiction $M, \mu \Vdash_g A$. Then we must have $\mathbf{t}A^{\mu} \in S$. This is impossible, since we have $\mathbf{f}A^{\mu} \in S$, by the definition of a consistency property.

So suppose that the claim holds for all signed prefixed formulas $\mathbf{u}A^{\mu}$ with the degree of A smaller than h; let $E = \mathbf{u}A^{\mu} \in S$ where h is the degree of A. Consider the form of E.

- $E = \mathbf{t}B \wedge C^{\mu}$. Then $S \cup \{\mathbf{t}B^{\mu}, \mathbf{t}C^{\mu}\} \in \mathbf{C}^*$ since $\mathbf{C}^*$ is a consistency property. So $\mathbf{t}B^{\mu} \in S$ and $\mathbf{t}C^{\mu} \in S$, since S is downward closed, hence maximal. By the induction hypothesis, $M, \mu \Vdash_g B$ and $M, \mu \Vdash_g C$. Thus $M, \mu \Vdash_g B \wedge C$. Analogous reasoning goes for $E = \mathbf{f}B \vee C^{\mu}$—and, for that matter, for $\mathbf{t}\neg B^{\mu}$ and $\mathbf{f}\neg B^{\mu}$.
- $E = \mathbf{t}B \vee C^{\mu}$. Then either $S \cup \{\mathbf{t}B^{\mu} \in \mathbf{C}^*$ or $S \cup \{\mathbf{t}C^{\mu}\} \in \mathbf{C}^*$. Since S is maximal either $\mathbf{t}B^{\mu} \in S$ or $\mathbf{t}C^{\mu} \in S$. Then by induction hypothesis, either $M, \mu \Vdash_g B$ or $M, \mu \Vdash_g C$. Thus we must have $M, \mu \Vdash_g B \vee C$. Analogous reasoning goes for $E = \mathbf{f}B \wedge C^{\mu}$.

- $E = \mathbf{t}\Box_i C^{\mu}$. Consider any ν such that $\mu \mathcal{R}_i \nu$. There are in principle two cases. For one, $\mu = \nu$ and $A(j)$ is KD, KDB, KD4, KD5 or KD45, $j \leq i$ by $\mathcal{N}$ and there is no $\nu \in G$ such that $S, \Sigma \rhd \mu/\nu : j$. But since S is downward saturated, there must be some $\mu\alpha$ with $S, \Sigma \rhd \mu/\mu\alpha : j$ and $\mathbf{t}C^{\mu\alpha} \in S$. Hence there is some world $\nu = \mu\alpha \in G$ such that $S, \Sigma \rhd \mu/\nu : j$. This is absurd. So then we must have the other case: $S, \Sigma \rhd \mu/\nu : i$. Therefore, since $\mathbf{C}^*$ is a consistency property, $S \cup \{\mathbf{t}C^{\nu}\} \in \mathbf{C}^*$. Since S is maximal, $\mathbf{t}C^{\nu} \in S$. Then by the induction hypothesis, $\mathcal{M}, \nu \Vdash_g C$. Since ν was arbitrary, we have established that $\mathcal{M}, \mu \Vdash_g \Box_i C$. Analogous reasoning goes for $E = \mathbf{f}\Diamond_i C^{\mu}$.

- $E = \mathbf{t}\Diamond_i C^{\mu}$. Since S is downward saturated, there is some $\mu\alpha$ such that $S, \Sigma \rhd \mu/\mu\alpha : i$ and $\mathbf{t}C^{\mu\alpha} \in S$. We have therefore that $\mu \mathcal{R}_i \mu\alpha$ (by definition of $\mathcal{R}_i$) and $\mathcal{M}, \mu\alpha \Vdash_g C$ (by induction hypothesis). Thus $\mathcal{M}, \mu \Vdash_g \Diamond_i C$. Analogous reasoning goes for $E = \mathbf{f}\Box_i C^{\mu}$.

- $E = \mathbf{t}\forall x C^{\mu}$. Let t be some element of $\mathcal{D}(\mu)$: that means $S, \Sigma \rhd t : \mu$ and either t is a constant or t occurs in S. So by definition of consistency property, $S \cup \{\mathbf{t}C[t/x]^{\mu}\} \in \mathbf{C}^*$—that is, $\mathbf{t}C[t/x]^{\mu} \in S$. Let g' be an x-variant of g such that $g(x) = t$. If g is the identity on parameters, so is g' (x is a variable). Thus by induction hypothesis $\mathcal{M}, \mu \Vdash_{g'} C[t/x]$; by Lemma 1 then $\mathcal{M}, \mu \Vdash_{g'} C$. This shows $\mathcal{M}, \mu \Vdash_g \forall x C$. Analogous reasoning goes for $E = \mathbf{f}\exists x C^{\mu}$.

- $E = \mathbf{t}\exists x C^{\mu}$. Since S is downward saturated, $\mathbf{t}C[c/x]^{\mu} \in S$ for some c with $S, \Sigma \rhd c : \mu$. Then $c \in \mathcal{D}(\mu)$; let g' be an x-variant of g with $g'(x) = c$. By induction hypothesis $\mathcal{M}, \mu \Vdash_{g'} C[c/x]$ and so by Lemma 1 $\mathcal{M}, \mu \Vdash_g C$. Thus $\mathcal{M}, \mu \Vdash_g \exists x C$. Analogous reasoning goes for $E = \mathbf{f}\forall x C^{\mu}$.

■

The results thus far are summarized in Proposition 14.

Proposition 14 *Let* $\mathbf{C}$ *be a first-order* $\mathcal{S}$*-consistency property for* Σ *fair (to* κ *and P) that is X-compatible where X is a set of sentences of* $L(CONST)$*. Let* $S \in \mathbf{C}$*, where S is a set of signed formulas from* $L(CONST)$ *labeled with the prefix* ε*. Then there is a k-modal model in which S is satisfiable.*

Proof. By Proposition 8, we can extend $\mathbf{C}$ to $\mathbf{C}^*$ an alternate consistency property of finite character; we still have $S \in \mathbf{C}^*$. In fact, we can divide P and κ into disjoint Q and R and disjoint ξ and ζ with Σ still fair to R and ζ, so that $S \in \mathbf{C}^*|^{Q,\xi}$. After all, S contains only $L(\text{CONST})$ sentences. Thus by Lemma 11 we can find $S' \in \mathbf{C}^*$ downward saturated with $S \subseteq S'$. By Lemma 13, S' is satisfiable in the term model for S' on any assignment g that is the identity function on parameters and any function θ that is the identity function on prefixes. Thus since $S \subseteq S'$, S is also satisfiable in the term model for S' with respect to g and θ. ■

2.5.3 Completeness Proper

We are now in a position to prove completeness immediately.

Theorem 2 (Completeness) *Suppose A is valid. Then there is a closed tableau for $\triangleright \mathbf{f}A^{\varepsilon}$.*

Proof. We have seen that we can construct a fair typing Σ for P and κ. Given a set Γ of signed prefixed sentences of $L(\mathrm{CONST}\cup P)^{\Pi(\kappa)}$, let Σ_G be $\{\mu/\nu \in \Sigma \mid \mu$ and ν occur in $\Gamma\} \cup \{p : \mu \in \Sigma \mid p$ occurs in $\Gamma\}$; say Γ is tableau-consistent if $\Sigma_G \triangleright \Gamma$ is a tableau line and there is no closed tableau for $\Sigma_G \triangleright \Gamma$. Let $\mathbf{C}$ be the collection of all such tableau-consistent sets. We claim $\mathbf{C}$ is an first-order S-consistency property.

Suppose $\Gamma \in \mathbf{C}$. Then if $\mathbf{t}A^{\mu} \in \Gamma$ and $\mathbf{f}A^{\mu} \in \Gamma$, we can apply *closure* to obtain a closed tableau, contrary to assumption. Likewise if $\mathbf{f}\top^{\mu} \in \Gamma$.

Again, for the remaining cases we have common reasoning. We suppose for the sake of argument that $\Gamma \in \mathbf{C}$ but $\Gamma \cup C \notin \mathbf{C}$ for appropriate C. That would mean there was a closed tableau T for $\Sigma_{\Gamma\cup C} \triangleright \Gamma \cup C$. By applying a tableau rule from $\Sigma_{\Gamma} \triangleright \Gamma$ to $\Sigma_{\Gamma\cup C} \triangleright \Gamma \cup C$, we construct a closed tableau for $\Sigma_{\Gamma} \triangleright \Gamma$ using T. This is impossible.

We consider representative cases. For $\mathbf{t}A \wedge B^{\mu} \in \Gamma$, suppose not $\Gamma' = \Gamma \cup \{\mathbf{t}A^{\mu}, \mathbf{t}B^{\mu}\} \notin \mathbf{C}$. Then there is a closed tableau T for $\Sigma_{\Gamma'} \triangleright \Gamma'$. Since $\Sigma_{\Gamma'} = \Sigma_{\Gamma}$ we can construct a closed tableau thus:

$$\cfrac{\Sigma_{\Gamma} \triangleright \Gamma}{\cfrac{\Sigma_{\Gamma} \triangleright \Gamma'}{T}}$$

We obtain essentially the same for $\mathbf{t}\neg A^{\mu} \in \Gamma$ (and for $\mathbf{f}A \vee B^{\mu}$, $\mathbf{f}\neg A^{\mu}$).

For $\mathbf{t}\forall x A^{\mu} \in \Gamma$, we suppose there is some $\Gamma' = \Gamma \cup \{\mathbf{t}A[p/x]^{\mu}\}$ where p is a constant or p occurs in Γ and $S, \Sigma \triangleright p : \mu$. But under these circumstances, $S, \Sigma_{\Gamma} \triangleright p : \mu$, so the tableau above meets the side conditions on the universal rule.

For $\mathbf{t}\exists x A^{\mu} \in \Gamma$, we suppose that for every c with $S, \Sigma \triangleright p : \mu$, $\Gamma \cup \{\mathbf{t}A[p/x]^{\mu}\} \notin \mathbf{C}$. But since Σ is fair and $\Sigma_{\Gamma} \triangleright \Gamma$ is a tableau line, there must be some such c that does not occur in Σ_{Γ}. Consider $\Gamma' = \Gamma \cup \{\mathbf{t}A[c/x]^{\mu}\}$: there must be a closed tableau T for this. We derive a contradiction by constructing a new closed tableau:

$$\cfrac{\Sigma_{\Gamma} \triangleright \Gamma}{\cfrac{\Sigma_{\Gamma}, c : \mu \triangleright \Gamma'}{T}}$$

The same goes, *mutatis mutandis*, for the modal cases.

Finally, for $\mathbf{t}A \vee B^{\mu} \in \Gamma$, suppose both $\Gamma' = \Gamma \cup \{\mathbf{t}A^{\mu}\} \notin \mathbf{C}$ and $\Gamma'' = \Gamma \cup \{\mathbf{t}B^{\mu}\} \notin \mathbf{C}$. Then we have tableaux T' for $\Sigma_{\Gamma'} \triangleright \Gamma'$ and T'' for $\Sigma_{\Gamma''} \triangleright \Gamma''$. Again $\Sigma_{\Gamma} = \Sigma_{\Gamma'} = \Sigma_{\Gamma''}$, so we construct

$$\dfrac{\Sigma_\Gamma \rhd \Gamma}{\dfrac{\Sigma_\Gamma \rhd \Gamma'}{T'} \qquad \dfrac{\Sigma_\Gamma \rhd \Gamma''}{T''}}$$

(We obtain essentially the same for $\mathbf{f}A \wedge B^\mu$.)

Having shown that $\mathbf{C}$ is a consistency property, suppose there is no closed tableau for $\rhd \mathbf{f}A^\varepsilon$. It follows that $\{\mathbf{f}A^\varepsilon\} \in \mathbf{C}$. Then by Proposition 14, there is a model $\mathcal{M}$ and world w with $\mathcal{M}, w \not\Vdash A$. ■

3 First-order Multi-Modal Herbrand Deduction

A major drawback to the use of the first-order tableau calculus studied in section 2 is the form of quantifier rules. Rules like the existential rule

$$\frac{\Sigma \rhd \Gamma, \mathbf{t}\exists x A^\mu}{\Sigma, c : \mu \rhd \Gamma, \mathbf{t}\exists x A^\mu, \mathbf{t}A[c/x]^\mu}$$

have a side condition that requires the parameter c to be new; this is known as an *eigenvariable* condition. Because of the eigenvariable condition, the order in which inferences are applied in tableau proofs matters. In particular, under appropriate circumstances, it will be possible to construct a closed tableau by applying the following inferences:

$$\dfrac{\dfrac{\Sigma \rhd \Gamma, \mathbf{t}\exists x A^\mu, \mathbf{t}\forall x B^\mu}{\Sigma, c : \mu \rhd \Gamma, \mathbf{t}\exists x A^\mu, \mathbf{t}A[c/x]^\mu, \mathbf{t}\forall x B^\mu}\,(\mathbf{t}\exists)}{\Sigma, c : \mu \rhd \Gamma, \mathbf{t}\exists x A^\mu, \mathbf{t}A[c/x]^\mu, \mathbf{t}\forall x B^\mu, \mathbf{t}B[c/x]^\mu}\,(\mathbf{t}\forall)$$
$$\vdots$$

However, suppose we construct a tree of tableau lines exactly corresponding to that tableau except that the existential and universal inferences are swapped, as below:

$$\dfrac{\dfrac{\Sigma \rhd \Gamma, \mathbf{t}\exists x A^\mu, \mathbf{t}\forall x B^\mu}{\Sigma \rhd \Gamma, \mathbf{t}\exists x A^\mu, \mathbf{t}\forall x B^\mu, \mathbf{t}B[c/x]^\mu}\,(\mathbf{t}\forall)}{\Sigma, c : \mu \rhd \Gamma, \mathbf{t}\exists x A^\mu, \mathbf{t}A[c/x]^\mu, \mathbf{t}\forall x B^\mu, \mathbf{t}B[c/x]^\mu}\,(\mathbf{t}\exists)$$
$$\vdots$$

This tree will not constitute a tableau, because the eigenvariable condition is not met at the existential inference. (As it happens, the typing side condition on the universal inference is not met either, but the same problem with the eigenvariable condition still arises in proof systems where universal inferences do not have such side conditions.) This asymmetry means that it may be necessary in proof search with this system to search not only for the right inferences but the right inferences in the right order.

In this section, we describe a standard method for reformulating quantifier rules in order to eliminate this asymmetry. This method calls for the use of *Herbrand terms* (also known, from another point of view, as Skolem terms) in place of fresh parameters for existential rules. Herbrand terms have the form $h(X)$; h is a symbol uniquely associated with some formula A in the base language which might serve as the principal of an existential rule; X is a tuple of terms. We refer to a proof system which calls for parameters to satisfy the eigenvariable condition as a *ground* tableau calculus; we refer to a proof system that calls for the use of Herbrand terms as a *Herbrand* tableau calculus.

The rationale behind the use of a Herbrand term $h(X)$ at an existential inference R goes like this. Regardless of the order in which inferences are applied in a closed tableau, there will be some parameters that must occur on the tableau line where R applies. For example, some parameters must appear on this tableau line as a result of the instantiations that must take place in deriving the principal expression of R. The terms X which are supplied as an argument to the Herbrand term $h(X)$ identify all these parameters indirectly. The structure $h(X)$ can therefore serve as a placeholder for a new parameter that is chosen to be different from each of the terms in X. The structure $h(X)$ thus packs all the information required to allow the inferences in the proof to be reordered and an appropriate parameter chosen so that the inference at A respects the eigenvariable condition.

In modal deduction, of course, eigenvariable conditions are not only associated with existential inferences; they are also associated with inferences of possibility and the special inference rules for serial modalities. Modal Herbrand inference therefore requires that we introduce Herbrand terms to describe transitions among possible worlds and Herbrand prefixes to name possible worlds, in addition to introducing first-order Herbrand terms to represent first-order parameters. In this case, the arguments X to Herbrand terms must mix first-order Herbrand terms and Herbrand prefixes, since logical formulas can encode dependencies among first-order and modal parameters.

3.1 Formalism

To describe the Herbrand inference system, we rework the definitions of section 2.3. We begin by assuming two countably infinite sets of symbols: a set H of *first-order Herbrand functions* and Υ of *modal Herbrand functions*. By mutual recursion, we can now define sets P_H of *first-order Herbrand terms* and κ_Υ of *modal Herbrand terms*:

Definition 29 (Herbrand terms and prefixes) *Let t_0 be a Herbrand prefix and let $t_1, \ldots, t_n$ be a sequence (possibly empty), where each t_i is either an element of CONST, a first-order Herbrand term, or a Herbrand prefix. Then if h is a first-order Herbrand function then $h(t_0, t_1, \ldots, t_n)$ is a* first-order Herbrand term. *If η is a modal Herbrand function then $\eta(t_0, t_1, \ldots, t_n)$ is a* modal Herbrand term. *A* Herbrand prefix *is any finite sequence of modal Hebrand terms.*

The terms that this definition provides can be named as a class $H = \mathrm{CONST} \cup P_H \cup \Pi(\kappa_\Upsilon)$. The basic expressions in proofs will now be prefixed formulas in the language $L(\mathrm{CONST} \cup P_H)^{\Pi(\kappa_\Upsilon)}$. The formulas continue to be *signed*; moreover, now they must also be *tracked* to indicate the sequence of instantiations that has taken place in the derivation of an expression.

Definition 30 (Signed, tracked expressions) *If E denotes the expressions of some class, then the* signed, tracked expressions *of that class are expressions of the form* $\mathbf{t}e_I$ *or* $\mathbf{f}e_I$ *where e is an expression of e and I is a finite sequence (possibly empty) of elements of H.*

We say that a signed tracked expression $\mathbf{u}e_I$ *tracks* a term t just in case t occurs as a subterm of some term in I.

It is clear that there are countably many first-order Herbrand terms, Herbrand prefixes, and formulas in $L(\mathrm{CONST} \cup P_H)$. We can therefore describe a correspondence as follows. If A is a formula of the form $\forall xB$ or $\exists xB$, we define a corresponding first-order Herbrand function h_A so that each first-order Herbrand function is h_A for some A. If A is a formula of the form $\Box_i B$ or $\Diamond_i B$, we define a corresponding modal Herbrand function η_A. If A is a formula of the form $\Box_i B$ and j is a modality, we also define a modal Herbrand function $\eta_{s(j,A)}$. And if A is any formula, μ is a Herbrand prefix and j is a modality, we define a modal Herbrand function $\eta_{e(j,\mu,A)}$. We insist that the sets $\{\eta_A\}$, $\{\eta_{s(j,A)}\}$ and $\{\eta_{e(j,\mu,A)}\}$ be disjoint and that their union be Υ. Now we have:

Definition 31 (Herbrand Typings) *A Herbrand typing for a language $L(CONST \cup P_H)$ (under a correspondence as just described) is a set Σ of statements, each of which takes one of two forms:*

1. $\mu/\mu\eta : i$ *where: μ is a Herbrand prefix and η is a modal Herbrand term meeting one of the following conditions:*
   - η *is* $\eta_A(\mu, I)$ *and A is $\Box_i B$ or $\Diamond_i B$.*
   - η *is* $\eta_{s(i,A)}(\mu, I)$.
   - η *is* $\eta_{e(i,\mu,A)}(I)$.
2. $t : \mu$ *where t is a first-order Herbrand term of the form $h(\mu, I)$.*

Note that the first bullet under clause 1 does not place any restriction on A beyond what η_A already requires, but simply accesses the modality i from A.

A sequence of modal and first-order Herbrand terms X determines a Herbrand typing Ξ_X, consisting of the appropriate $\mu/\mu\eta : i$ for each modal Herbrand term η that occurs in X (possibly as a subterm) and the appropriate $h : \mu$ for each first-order Herbrand term h that occurs in X (possibly as a subterm).

A Herbrand typing *for a set or multiset* Γ *of signed, tracked expressions of* $L(CONST \cup P_H)^{\Pi(\kappa_\Upsilon)}$ *whenever* Σ *is a finite Herbrand typing that contains an appropriate expression* $t : \mu$ *for each first-order Herbrand term that occurs in* Γ *or in* Σ *itself and an appropriate expression* $\mu/\mu\eta : i$ *for each modal Herbrand term* η *that occurs in* Γ *or in* Σ *itself.*

The definition of derivations of typing judgments carries over from Definition 14 to Herbrand typings unchanged.

A Herbrand multi-modal tableau line is an expression of the form $\Sigma \triangleright \Gamma$ where Γ is a finite multiset of signed, tracked expressions of $L(\text{CONST} \cup P_H)^{\Pi(\kappa_\Upsilon)}$ and Σ is a finite Herbrand typing. (Σ need not be a typing for Γ.)

Definition 32 (Tableau rule) *For first-order multi-modal Herbrand deductions over a regime* S*, we will use the following tableau rules:*

1. *closure—with A an atomic formula and subject to the side condition that* $\Xi_X \subseteq \Sigma$ *and (for the binary rule)* $\Xi_Y \subseteq \Sigma$*:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}A^{\mu}_{X}, \mathbf{f}A^{\mu}_{Y}}{\bot} \qquad \frac{\Sigma \triangleright \Gamma, \mathbf{f}\top^{\mu}_{X}}{\bot}$$

2. *conjunctive:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}A \wedge B^{\mu}_{X}}{\Sigma \triangleright \Gamma, \mathbf{t}A \wedge B^{\mu}_{X}, \mathbf{t}A^{\mu}_{X}, \mathbf{t}B^{\mu}_{X}} \qquad \frac{\Sigma \triangleright \Gamma, \mathbf{f}A \vee B^{\mu}_{X}}{\Sigma \triangleright \Gamma, \mathbf{f}A \vee B^{\mu}_{X}, \mathbf{f}A^{\mu}_{X}, \mathbf{f}B^{\mu}_{X}}$$

3. *disjunctive:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{f}A \wedge B^{\mu}_{X}}{\Sigma \triangleright \Gamma, \mathbf{f}A \wedge B^{\mu}_{X}, \mathbf{f}A^{\mu}_{X} \qquad \Sigma \triangleright \Gamma, \mathbf{f}A \wedge B^{\mu}_{X}, \mathbf{f}B^{\mu}_{X}}$$

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}_{X}}{\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}_{X}, \mathbf{t}A^{\mu}_{X} \qquad \Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}_{X}, \mathbf{t}B^{\mu}_{X}}$$

4. *negation:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}}{\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{f}A^{\mu}_{X}} \qquad \frac{\Sigma \triangleright \Gamma, \mathbf{f}\neg A^{\mu}_{X}}{\Sigma \triangleright \Gamma, \mathbf{f}\neg A^{\mu}_{X}, \mathbf{t}A^{\mu}_{X}}$$

5. *possibility—where* η *is* $\eta_A(\mu, X)$ *for* $\mathbf{u}A^{\mu}_{X}$ *the principal of the rule (either* $\Box_i A$ *or* $\Diamond_i A$*):*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{f}\Box_i A^{\mu}_{X}}{\Sigma, \mu/\mu\eta : i \triangleright \Gamma, \mathbf{f}\Box_i A^{\mu}_{X}, \mathbf{f}A^{\mu\eta}_{X,\mu\eta}} \qquad \frac{\Sigma \triangleright \Gamma, \mathbf{t}\Diamond_i A^{\mu}_{X}}{\Sigma, \mu/\mu\eta : i \triangleright \Gamma, \mathbf{t}\Diamond_i A^{\mu}_{X}, \mathbf{t}A^{\mu\eta}_{X,\mu\eta}}$$

6. *necessity—subject to the side condition that there is a typing derivation* $S, \Xi_{X,N,\nu} \triangleright \mu/\nu : i$ *for a sequence N of elements of H:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}\Box_i A^{\mu}_X}{\Sigma \triangleright \Gamma, \mathbf{t}\Box_i A^{\mu}_X, \mathbf{t}A^{\nu}_{X,N,\nu}} \qquad \frac{\Sigma \triangleright \Gamma, \mathbf{f}\Diamond_i A^{\mu}_X}{\Sigma \triangleright \Gamma, \mathbf{f}\Diamond_i A^{\mu}_X, \mathbf{f}A^{\nu}_{X,N,\nu}}$$

7. *special necessity—subject to the side condition that* $A(i)$ *is one of KD, KDB, KD4, KD5 or KD45, that* $i \leq j$ *according to* N *, that* η *is* $\eta_{s(i,A)}(\mu, X)$ *for* $\mathbf{u}A^{\mu}_X$ *the principal of the rule:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}\Box_j A^{\mu}_X}{\Sigma, \mu/\mu\eta : i \triangleright \Gamma, \mathbf{t}\Box_j A^{\mu}_X, \mathbf{t}A^{\mu\eta}_{X,\mu\eta}} \qquad \frac{\Sigma \triangleright \Gamma, \mathbf{f}\Diamond_j A^{\mu}_X}{\Sigma, \mu/\mu\eta : i \triangleright \Gamma, \mathbf{f}\Diamond_j A^{\mu}_X, \mathbf{f}A^{\mu\eta}_{X,\mu\eta}}$$

8. *extra special necessity—subject to the side conditions that* $A(i)$ *is one of KD, KDB, KD4, KD5 or KD45, that* $A(j)$ *is one of K5, K45, KD5, KD45 or S5, that* $i \leq j$ *according to* N *, that there is a derivation* $S, \Xi_{X,N} \triangleright \mu/\nu : j$ *for a sequence N of elements of H, that* η *is* $\eta_{e(i,\mu,A)}(\nu, X, N)$*:*

$$\frac{\Sigma, \mu/\mu\eta : i \triangleright \Gamma, \mathbf{u}A^{\nu}_X}{\Sigma, \mu/\mu\eta : i \triangleright \Gamma, \mathbf{t}\top^{\mu\eta}_{X,N,\mu\eta}, \mathbf{u}A^{\nu}_X}$$

9. *existential—subject to the side condition that h is* $h_A(\mu, X)$ *where* $\mathbf{u}A^{\mu}_X$ *is the principal of the rule:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}\exists x A^{\mu}_X}{\Sigma, h : \mu \triangleright \Gamma, \mathbf{t}\exists x A^{\mu}, \mathbf{t}A[h/x]^{\mu}_{X,h}} \qquad \frac{\Sigma \triangleright \Gamma, \mathbf{f}\forall x A^{\mu}_X}{\Sigma, h : \mu \triangleright \Gamma, \mathbf{f}\forall x A^{\mu}_X, \mathbf{f}A[h/x]^{\mu}_{X,h}}$$

10. *universal—subject to the side condition that there is a typing derivation* $S, \Xi_{X,N,t} \triangleright t : \mu$ *for a sequence N of elements of H:*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}\forall x A^{\mu}_X}{\Sigma \triangleright \Gamma, \mathbf{t}\forall x A^{\mu}_X, \mathbf{t}A[t/x]^{\mu}_{X,N,t}} \qquad \frac{\Sigma \triangleright \Gamma, \mathbf{f}\exists x A^{\mu}_X}{\Sigma \triangleright \Gamma, \mathbf{f}\exists x A^{\mu}_X, \mathbf{f}A[t/x]^{\mu}_{X,N,t}}$$

Tableaux, branches, agreement and closure remain as in Definition 18 and following.

Remark. Once again the distinction between serial and non-serial modalities and the presence of euclidean modalities leads to some surprises in the definitions.

One surprise is the side condition governing typings at the closure rules. This is required to correctly model modal operators with accessibility relations that may be empty. With non-serial modalities, it becomes possible to construct a Herbrand prefix $\mu\eta$ for which no corresponding world exists in some model, even though μ does correspond to a world in that model. In this case it would be incorrect to reason about world $\mu\eta$. Such cases cannot be distinguished from cases where $\mu\eta$ does correspond to a world in the model by inspection of a typing Ξ_X where X includes η. Instead, we must check whether the worlds that have been explicitly introduced by possibility inferences (and special or extra special necessity inferences) include $\mu\eta$.

Another surprise is the extension of the terms tracked on universal and necessity tableau rules by a parametric list N of additional elements of H. The reason for this, informally, is that we may have situations in which we can only derive $\mu/\nu : i$ by accessing a prefix μ' some of whose elements occur neither in μ nor in ν. For example, consider a regime S defined by

$$\begin{array}{l} \langle A = \{0 \mapsto K, 1 \mapsto K5, 2 \mapsto K5, 3 \mapsto K4\}, \\ N \ = \{0 \leq 1, 0 \leq 2, 0 \leq 3, 1 \leq 3, 2 \leq 3\}, \\ Q \ = \mathit{constant}\rangle \end{array}$$

Then from $/\alpha : 0, /\beta : 1, \gamma : 2$ we can derive $\beta/\gamma : 3$ but without $/\alpha : 0$ we cannot derive this. (The derivation first shows $\beta/\alpha : i$ via inclusion and euclideanness, then shows $\alpha/\gamma : j$ via inclusion and euclideanness, and finally $\beta/\gamma : k$ via inclusion and transitivity.)

This characteristic of typing derivations translates into corresponding facts about closed Herbrand tableaux. For example we have that the set $\{\Diamond_1 \Box_3 q, \Diamond_2 \neg q\}$ is S-consistent, but the set $\{\Diamond_0 p, \Diamond_1 \Box_3 q, \Diamond_2 \neg q\}$ is not. The derivation of inconsistency is obtained by introducing transitions $/\alpha : 0, /\beta : 1, \gamma : 2$ corresponding to the three possibility statements, then instantiating $\Box_3 q^\beta$ to show q^γ. In this case, any further use of q^γ in fact depends on having introduced world α already. So were we to rewrite this proof using Herbrand terms, we must encode this by creating a tracked formula as $q^\gamma_{\beta,\alpha,\gamma}$. This accounts for the tracking of new terms for typing derivations on universal tableau rules. ■

Definition 33 (Proof) *A* Herbrand proof *is a closed Herbrand tableau for* $\triangleright\Gamma$.

The Herbrand calculus is also a sound and complete characterization of first-order modal models. In contrast to the semantic methods we used in section 2, we will establish this correctness result by syntactic methods, which relate Herbrand proofs to closed ground tableaux. Suppose Γ contains sentences of $L(\text{CONST})$ (prefixed by ε). Then the soundness theorem says that if there is a Herbrand proof of $\triangleright\Gamma$, then there is a closed first-order tableau for $\triangleright\Gamma$. The completeness theorem says that if there is a closed first-order tableau for $\triangleright\Gamma$, then there is a Herbrand proof of $\triangleright\Gamma$. In addition to the syntactic formulation of these theorems, there is another major reversal from our earlier results: now soundness is the difficult thing to show, whereas completeness is relatively straightforward. Section 3.3 presents the soundness result drawing on background introduced in Section 3.2. Finally, Section 3.4 proves completeness.

3.2 Background

Our syntactic methods for reasoning about tableaux exploit *permutability of inference* in the Herbrand tableau calculus. To develop the notion of permutability of inference, we need to make some observations about the tableau rules of Definition 32. We begin with those observations that are common to tableau calculi in general. Here as earlier, we can distinguish the principal and side expressions of each tableau inference (except closure). In each denominator tableau line, the occurrence of the principal expression and the side expression all derive from—or as we shall say, *originate in*—the occurrence of the principal in the numerator tableau line. Similarly, each of the remaining expressions in the denominator tableau line *originate in* an occurrence of the same expression in the numerator tableau line. (Here, as in [Kleene, 1951], we are assuming an *analysis* of each inference to specify this correspondence in the case where the same expression occurs multiple times in the numerator or denominator tableau line.)

Consider then two distinct tableau lines on the same branch b of a tableau. We will identify the line O closer to the root of the tableau as the *original* line and the the line further from the root of the tableau line as the *derivative* line, or the *derivative* line to O. Applying the notion *orginates* transitively to expressions separated by multiple steps of inference along a branch, we can say that each expression occurrence in the derivative line *originates in* a unique expression occurrence in the original line. By extension, when an inference L *applies at* the derivative line (meaning that the line is the numerator of L), and so the principal formula of L originates in some expression E of the original line, we say that L itself originates in E. Call the inference that applies at the original line O; in the more specific case that L originates in a side expression of the denominator of O on the branch b, we say that inference L originates *in* inference O.

Now, in the case of the Herbrand tableau calculus in particular, we distinguish the *possibility*, *special necessity*, *extra special necessity* and *existential* rules as *Herbrand rules*, since any occurrence of these inferences in a tableau is associated with some Herbrand term x that is *introduced* there. Conversely, we distinguish the *necessity*, *extra special necessity* and *universal* rules as *general* rules where there is the possibility of *introducing* a *general* term x either as the value for a variable or as an element of a sequence N of terms introduced for the purposes of typing.

The form of general rules is such that at a general inference L, any side expression of L tracks the term x that L introduces. We extend the terminology of tracking to describe inferences: when a side expression of an inference L tracks x, we say that L tracks x. In the case of a closure inference from $\mathbf{f}\top^{\mu}_{X}$, we say the inference tracks t just in case t occurs as a subterm in some term in X; for a closure inference from $\mathbf{t}E_X$ and $\mathbf{f}E_Y$, the inference tracks t just in case t occurs as a subterm in some term in X or in Y. As we consider the tracking of terms in tableaux more broadly, we discover that if inference L originates in inference O, and O tracks x, then L tracks x—This follows from a simple induction on the length of the path from O to L and the observation that tracked terms on principal expressions are always preserved on side expressions.

Interchanges of inference are transformations on proofs. They appeal to the two basic operations of *contraction* and *weakening*, which must be cast as transformations on proofs in this framework. (In other proof systems, contraction and weakening are introduced as explicit *structural rules*.)

Lemma 15 (Weakening (by formulas)) *Let T be a Herbrand tableau and let Δ be a finite multiset of signed, tracked prefixed formulas (in the same language as T). Denote by $T+\Delta$ a structure with nodes and edges exactly like T, but where any node in T carries $\Sigma \triangleright \Gamma$, the corresponding node in $T+\Delta$ carries $\Sigma \triangleright \Gamma, \Delta$; otherwise corresponding nodes in T and $T+\Delta$ both carry $\bot$. Then $T+\Delta$ is a Herbrand tableau and $T+\Delta$ is closed just in case T is closed.*

Lemma 16 (Weakening (by typings)) *Let T be a Herbrand tableau and let Φ be a finite Herbrand typing (in the same language as T). Denote by $T+\Phi$ a structure with nodes and edges exactly like T, but where any node in T carries $\Sigma \triangleright \Gamma$, the corresponding node in $T+\Phi$ carries $\Sigma, \Phi \triangleright \Gamma$; otherwise corresponding nodes in T and $T+\Phi$ both carry $\bot$. Then $T+\Phi$ is a Herbrand tableau and $T+\Phi$ is closed just in case T is closed.*

Lemma 17 (Contraction) *Let T be a Herbrand tableau whose root carries $\Sigma \triangleright \Gamma, E, E$. Then we can construct a tableau T' whose root carries $\Sigma \triangleright \Gamma, E$, where T' is closed if and only if T is, where the height of T' is at most the height of T and where there is a one-to-one correspondence (also preserving order of inferences) that takes any inference of T' to an inference with the same principal and side expressions in T.*

These lemmas follow from straightforward induction on the structure of tableaux.

To define interchanges of inference, consider two inferences O and D on the same branch in a tableau T, with O the original and D the derivative. We say that O and D are *adjacent* when the numerator of D is in fact a denominator of O. Suppose O and D are adjacent, and D does not originate *in* O; then rooted at O in T we have the following scenario (although of course D may apply to any denominator of O):

$$\dfrac{\Sigma \triangleright \Gamma}{\dfrac{\Psi}{T_{1D} \ldots T_{mD}}\, D \qquad T_{2O} \ldots T_{nO}}\, O$$

Then the tableau T' *interchanges* O and D in T if T' is exactly like T except that the order of O and D along the branch has changed. Explicitly, such a T' must consist of T without the subtree rooted at O; in place of this subtree, T' must include a new subtree rooted with an instance of D, with adjacent instances of O below, and for the subtrees below copies (possibly weakened) of $T_{1D}, \ldots, T_{mD}$ and $T_{2O}, \ldots, T_{nO}$. Since the leaves of T' are then just the leaves of T, T' is closed if and only if T is.

We can now present the basic result about inference in classical Herbrand inference systems.

Lemma 18 (Interchange) *Suppose T is a Herbrand tableau and O and D are adjacent inferences in T neither of which originates in the other. Then there is a tableau T' that interchanges O and D in T.*

Proof. The proof consists of a simple analysis of the possible cases. The common reasoning behind these many cases shows how interchanges can be accomplished according to a few general schemas that describe how expressions should be propagated through tableau lines. We illustrate these schemas here with a case study of the positive tableau rules for $\vee$ and $\neg$; the schemas can also be presented in abstract generality, as in [Kleene, 1951], in a more rigorous (but perhaps less perspecuous) style.

First, we consider the interchange of two positive $(\neg)$ inferences; we start with the following configuration:

$$\cfrac{\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{t}\neg B^{\nu}_{Y}}{\cfrac{\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{f}A^{\mu}_{X}, \mathbf{t}\neg B^{\nu}_{Y}}{\begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{f}A^{\mu}_{X}, \mathbf{t}\neg B^{\nu}_{Y}, \mathbf{f}B^{\nu}_{Y}\\ T\end{array}}}$$

Interchanging the inferences can be accomplished by adjusting the intermediate tableau line to carry $\mathbf{f}B^{\nu}_{Y}$—the side expression of the former derivative inference—in place of $\mathbf{f}A^{\mu}_{X}$—the side expression of the former original inference:

$$\cfrac{\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{t}\neg B^{\nu}_{Y}}{\cfrac{\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{t}\neg B^{\nu}_{Y}, \mathbf{f}B^{\nu}_{Y}}{\begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{f}A^{\mu}_{X}, \mathbf{t}\neg B^{\nu}_{Y}, \mathbf{f}B^{\nu}_{Y}\\ T\end{array}}}$$

Next, suppose we start with an original positive $(\neg)$ inference and a derivative positive $(\vee)$, thus:

$$\cfrac{\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{t}B \vee C^{\nu}_{Y}}{\cfrac{\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{f}A^{\mu}_{X}, \mathbf{t}B \vee C^{\nu}_{Y}}{\begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{f}A^{\mu}_{X}, \mathbf{t}B \vee C^{\nu}_{Y}, \mathbf{t}B^{\nu}_{Y}\\ T'\end{array} \qquad \begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{f}A^{\mu}_{X}, \mathbf{t}B \vee C^{\nu}_{Y}, \mathbf{t}C^{\nu}_{Y}\\ T''\end{array}}}$$

The tableau that interchanges these inferences is:

$$\dfrac{\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{t}B \vee C^{\nu}_{Y}}{\dfrac{\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{t}B \vee C^{\nu}_{Y}, \mathbf{t}B^{\nu}_{Y}}{\begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{f}A^{\mu}_{X}, \mathbf{t}B \vee C^{\nu}_{Y}, \mathbf{t}B^{\nu}_{Y} \\ T'\end{array}} \qquad \dfrac{\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{t}B \vee C^{\nu}_{Y}, \mathbf{t}C^{\nu}_{Y}}{\begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{f}A^{\mu}_{X}, \mathbf{t}B \vee C^{\nu}_{Y}, \mathbf{t}C^{\nu}_{Y} \\ T''\end{array}}}$$

Now suppose we start with an original positive $(\vee)$ inference and a derivative positive $(\neg)$, thus:

$$\dfrac{\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{t}B \vee C^{\nu}_{Y}}{\dfrac{\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{t}B \vee C^{\nu}_{Y}, \mathbf{t}B^{\nu}_{Y}}{\begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{f}A^{\mu}_{X}, \mathbf{t}B \vee C^{\nu}_{Y}, \mathbf{t}B^{\nu}_{Y} \\ T'\end{array}} \qquad \begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{t}B \vee C^{\nu}_{Y}, \mathbf{t}C^{\nu}_{Y} \\ T''\end{array}}$$

Interchanging these inferences for the first time calls for weakening:

$$\dfrac{\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{t}B \vee C^{\nu}_{Y}}{\dfrac{\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{f}A^{\mu}_{X}, \mathbf{t}B \vee C^{\nu}_{Y}}{\begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{f}A^{\mu}_{X}, \mathbf{t}B \vee C^{\nu}_{Y}, \mathbf{t}B^{\nu}_{Y} \\ T'\end{array} \qquad \begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{f}A^{\mu}_{X}, \mathbf{t}B \vee C^{\nu}_{Y}, \mathbf{t}C^{\nu}_{Y} \\ T'' + \mathbf{f}A^{\mu}_{X}\end{array}}}$$

In the case where the unary rule is a Herbrand rule, the schema also calls for weakening T'' by any new typings that the rule introduces.

A final representative is provided by the interchange of two positive $\vee$ inferences. Initially, then, we have:

$$\dfrac{\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\nu}_{Y}, \mathbf{t}C \vee D^{\mu}_{X}}{\dfrac{\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\nu}_{Y}, \mathbf{t}A^{\nu}_{Y}, \mathbf{t}C \vee D^{\mu}_{X}}{\begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\nu}_{Y}, \mathbf{t}A^{\nu}_{Y}, \mathbf{t}C \vee D^{\mu}_{X}, \mathbf{t}C^{\mu}_{X} \\ T'\end{array} \quad \begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\nu}_{Y}, \mathbf{t}A^{\nu}_{Y}, \mathbf{t}C \vee D^{\mu}_{X}, \mathbf{t}D^{\mu}_{X} \\ T''\end{array}} \quad \begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\nu}_{Y}, \mathbf{t}B^{\nu}_{Y}, \mathbf{t}C \vee D^{\mu}_{X} \\ T'''\end{array}}$$

This interchange requires not only weakening but also the copying of the tableau T''':

$$\dfrac{\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\nu}_{Y}, \mathbf{t}C \vee D^{\mu}_{X}}{\dfrac{\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\nu}_{Y}, \mathbf{t}C \vee D^{\mu}_{X}, \mathbf{t}C^{\mu}_{X}}{\begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\nu}_{Y}, \mathbf{t}A^{\nu}_{Y}, \\ \mathbf{t}C \vee D^{\mu}_{X}, \mathbf{t}C^{\mu}_{X} \\ T'\end{array} \quad \begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\nu}_{Y}, \mathbf{t}B^{\nu}_{Y}, \\ \mathbf{t}C \vee D^{\mu}_{X}, \mathbf{t}C^{\mu}_{X} \\ T''' + \mathbf{t}C^{\mu}_{X}\end{array}} \qquad \dfrac{\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\nu}_{Y}, \mathbf{t}C \vee D^{\mu}_{X}, \mathbf{t}D^{\mu}_{X}}{\begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\nu}_{Y}, \mathbf{t}A^{\nu}_{Y}, \\ \mathbf{t}C \vee D^{\mu}_{X}, \mathbf{t}D^{\mu}_{X} \\ T''\end{array} \quad \begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\nu}_{Y}, \mathbf{t}B^{\nu}_{Y}, \\ \mathbf{t}C \vee D^{\mu}_{X}, \mathbf{t}D^{\mu}_{X} \\ T''' + \mathbf{t}D^{\mu}_{X}\end{array}}}$$

Given these schemas for interchanges—which classify interchanges based on the number of denominators in the two inferences to be exchanged—any particular interchange can be established by showing that any side conditions on the application of the inference rule continue to hold in the transformed proof. But, as is easily verified by inspection of Definition 32, the side conditions on Herbrand tableau rules depend only on the form of the principal and side expressions, the modal regime S and some typing Ξ_S. And the principal and side expressions of the inference, the typing Ξ_S, and of course the modal regime S are unchanged from the original tableau to the transformed tableau. ■

3.3 Soundness

The idea behind the soundness of any Herbrand tableau calculus is that the structure of Herbrand terms provides enough information to reconfigure a tableau (by an inductive process of interchanges of inference) so that equivalents of the eigenvariable conditions are enforced. For the modal Herbrand tableau calculus of Definition 32, it is easiest to perform this reconfiguration in steps. The first step ensures that no branch contains a *general* inference for any term x that occurs closer to the root than a *Herbrand inference that introduces* x. The next step simplifies the proof to eliminate duplicate Herbrand inferences and extraneous general inferences from any branch. Finally, we rewrite the simplified reordered proof in terms of the tableau rules of the first-order tableau calculus.

We begin by describing a relation $<$ on inferences in any Herbrand tableau $\mathcal{T}$.

Definition 34 *Let R and R' be two inferences in a Herbrand tableau $\mathcal{T}$. Then $R < R'$ if*

1. *R' is distinct from R, and R' originates in R*
2. *R is a Herbrand rule introducing x, and R' is a general rule with instance x*
3. *There is an inference R'' with $R < R''$ and $R'' < R'$.*

Lemma 19 *$<$ is a transitive, asymmetric relation.*

Proof. Clause 3 directly ensures that $<$ is transitive. It remains to show that we never have both $R < R'$ and $R' < R$. First, observe that if $R < R'$ and R tracks x then R' tracks x. This follows inductively from the definition of $<$. If R' originates in R then R' must track x by preservation. If R is a Herbrand inference, then every term x that R tracks is a subterm of the Herbrand term h that R introduces. Since R' tracks h, R' tracks x. Then the transitive case is derived by applying the induction hypothesis to R'' and then R'. Thus, if $R < R'$ and $R' < R$ then R tracks x just in case R' tracks x.

Now, we claim that if $R < R'$ and R tracks x just in case R' tracks x, then R' orginates in R and $R \neq R'$. We show this inductively from the definition of $<$. The first case is trivial: the case is just that R' originates in R and $R \neq R'$. Next, suppose R is a Herbrand inference introducing h and R' is a general inference with instance h. Then R' tracks h but R cannot track h because h contains as a proper subterm every term that R tracks. So this case is impossible. Then the transitive case follows by induction hypothesis and the transitivity of the originates relation.

If $R < R'$ and $R' < R$, we have $R < R$ by transitivity, and certainly R tracks x if and only if R tracks x. Therefore by the previous argument $R \neq R$. This is impossible. ■

Lemma 20 (Introduction) *Any* Herbrand proof $\mathcal{T}$ *may be transformed into another Herbrand proof* $\mathcal{T}'$ *by interchanges of inference with the following property: for any pair of inferences* R *and* R' *on a branch in* $\mathcal{T}'$*, if* $R < R'$ *then* R *is the original inference and* R' *is the derivative inference.*

Proof. Given a Herbrand tableau $\mathcal{T}$, say an inference R' is *misplaced* if there is another inference R with $R < R'$, where R' is the original inference and R is the derivative inference. With this terminology, the tableau $\mathcal{T}'$ we want is simply a Herbrand proof with no misplaced inferences. We will show that $\mathcal{T}$ can be transformed by interchanges of inference in such a way as to eliminate misplaced inferences. The argument is by induction on the *number* of misplaced inferences in the proof; we show how to transform $\mathcal{T}$ with $n+1$ misplaced inferences into $\mathcal{T}'$ with n misplaced inferences.

Since $\mathcal{T}$ is finite, it must contain a misplaced inference M with the property that no other misplaced inference applies to a tableau line derivative from M. Let $\mathcal{T}_M$ be the subproof of $\mathcal{T}$ rooted at M; we will construct a subproof $\mathcal{T}'_M$ without misplaced inferences from $\mathcal{T}_M$ by interchanges of inference. The tableau $\mathcal{T}'$ with n misplaced inferences that we need is then obtained from $\mathcal{T}$ by replacing $\mathcal{T}_M$ with $\mathcal{T}'_M$. The proof of the lemma is thereby reduced to the proof of the following proposition.

Proposition 21 *Let* $\mathcal{T}_M$ *be a Herbrand tableau in which only the root inference is misplaced.* $\mathcal{T}_M$ *can be transformed into a Herbrand tableau* $\mathcal{T}'_M$ *without misplaced inferences by interchanges of inference.*

We will call a Herbrand tableau in which only the root inference is misplaced a *penultimate* tableau. By the *height* of an inference G in $\mathcal{T}_M$, we mean the number of inferences that intervene on the branch from the root to G. Let the *degree* of a penultimate tableau $\mathcal{T}_M$ rooted with inference M be the sum of the heights of the inferences L in $\mathcal{T}_M$ for which $L < M$. Since $M \not< M$, a penultimate Herbrand tableau of degree zero has no misplaced inferences. Now we assume the proposition true for all penultimate Herbrand tableaux of degree d or less, and consider a penultimate tableau $\mathcal{T}_M$ of degree $d+1$ rooted in inference M. At least one subproof rooted at a denominator of M must contain an inference L with $L < M$. Call the adjacent inference to M in this subproof D.

Observe that D cannot originate in M. For if it did, we would have $M < D$, and hence by transitivity $L < D$, and by asymmetry $L \neq D$. But since L and D are on a common branch that means D is misplaced, contradicting our assumption on M. This means that we can interchange M and D according to the schemas of Lemma 18. After the interchange, the new subproofs rooted at M continue to have only M misplaced, but now must have lower degree, since the height of each inference above

D has been reduced by one. An illustration will make this point obvious; say M and D are binary inferences:

$$\dfrac{\dfrac{T' \qquad T''}{\Phi'}\,D \qquad\qquad T'''}{\Phi}\,M$$

We interchange M and D as follows:

$$\dfrac{\dfrac{T' \qquad T'''+}{\Phi''}\,M \qquad\qquad \dfrac{T'' \qquad T'''+}{\Phi'''}\,M}{\Phi}\,D$$

The subproofs rooted at M will consist of inferences from T''' whose height remains the same plus inferences from T' or T'' whose height is reduced by one. Thus we can apply the induction hypothesis to obtain new subproofs. We obtain T_R' by recombining the new subproofs using the D inference.

This completes the proof of the proposition and the lemma. ■

The next step is to ensure that the Herbrand inferences that apply along any branch do not have identical principal expressions.

Lemma 22 *A Herbrand proof T may be transformed into another Herbrand proof T' with two properties: there is a one-to-one correspondence (also preserving order of inferences) that takes any inference of T' to an inference with the same principal and side expressions in T; and if R and R' are two Herbrand inferences on the same branch in T' then the side expressions of R and R' are distinct.*

Proof. By induction on the number of pairs of Herbrand inferences R and R' that apply on a common branch with identical side expressions. If there is no such pair, we can use T as T'. So suppose the claim true for tableaux with n pairs or fewer, and consider a proof with $n+1$ pairs. Consider any pair R and R'; let R be the original and R' the derivative, and consider the tableau line that serves as the denominator of R'. This line must take the form $\Sigma \rhd \Gamma, E, E$ where E is the side expression of R'; one E derives from R' and given the preservation of formulas in tableau lines, another E must derive from the side expression of R. Therefore we can apply the contraction lemma to the subproof T_D rooted at the denominator of R' to obtain a tableau whose root carries $\Sigma \rhd \Gamma, E$. But this is the line to which R' applies; therefore we can use T_D in place of the original subproof rooted at R. The result contains at most n bad pairs. ■

As we shall see, these lemmas are enough to guarantee that when a Herbrand rule applies, the term h that it introduces is new to the sequent. We still need to ensure, however, that Herbrand terms are only used after being introduced; this requires a final tranformation on proofs.

Definition 35 (essential) *Let T be a Herbrand proof with T_L as a subproof, and let $\Sigma \rhd \Gamma$ be the line that the root of T_L carries. An expression occurrence E in Γ is* essential *if there is a closure inference in T_L that tracks every t that E tracks. Likewise, a typing statement* $\mu/\mu\eta : i$ *in* Σ *is* essential *if there is a closure instance in T_L that tracks* η *and a typing statement* $h : \mu$ *in* Σ *is* essential *if there is a closure instance in T_L that tracks h.*

Given a tableau T_L rooted with $\Sigma \triangleright \Gamma$, we introduce the following notation:

$$\Sigma_E \text{ for } \{E \in \Sigma \mid E \text{ is essential }\}$$
$$\Gamma_E \text{ for } \{E \in \Gamma \mid E \text{ is essential }\}$$

Lemma 23 (essential) *Suppose T is a Herbrand proof. Then T may be transformed into another Herbrand proof T' with two properties: there is a one-to-one correspondence (also preserving order of inferences) that takes any inference of T' to an inference with the same principal and side expressions in T; and whenever an inference L applies in T, in some denomator of L either the side expression of L or the typing introduced by L is essential.*

Proof. We describe T' as required by induction on the structure of tableaux; we construct T' so that if the root of T' carries $\Sigma \triangleright \Gamma$, every typing in Σ and expression in Γ is essential—this is the same tableau line as $\Sigma_E \triangleright \Gamma_E$.

Suppose T consists of an application of the closure rule

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}A^{\mu}_{X}, \mathbf{f}A^{\mu}_{Y}}{\perp}$$

Clearly $\mathbf{t}A^{\mu}_{X}$ and $\mathbf{f}A^{\mu}_{Y}$ are essential here: any term they track, they track. So it suffices to show the side conditions are met, namely $\Xi_X \subseteq \Sigma_E$ and $\Xi_Y \subseteq \Sigma_E$. Take Ξ_X; by definition, it consists of the appropriate $\mu/\mu\eta : i$ for each modal Herbrand term η that occurs in X (possibly as a subterm) and the appropriate $h : \mu$ for each first-order Herbrand term h that occurs in X (possibly as a subterm). Each such expression occurs in Σ, since we start from a closure inference, and is clearly essential in the line. In sum, then, we construct the closure inference:

$$\frac{\Sigma_E \triangleright \Gamma_E, \mathbf{t}A^{\mu}_{X}, \mathbf{f}A^{\mu}_{Y}}{\perp}$$

The same reasoning goes for the $\top$ closure inference.

Suppose the hypothesis holds for closed tableaux of height h, and consider a tableau of height $h+1$. We can construct a revised tableau T' by case analysis on the inference that applies at the root of T. The case for positive $\vee$ illustrates the complexities involved in any of these cases. We start from

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}_{X}}{\begin{array}{cc} \Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}_{X}, \mathbf{t}A^{\mu}_{X} & \Sigma \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}_{X}, \mathbf{t}B^{\mu}_{X} \\ T_1 & T_2 \end{array}}$$

We apply the induction hypothesis to the two subproofs, to obtain a derivation T_1' for $\Sigma_{1E} \triangleright \Gamma_{1E}$ and a derivation T_2' for $\Sigma_{2E} \triangleright \Gamma_{2E}$. (These tableau lines need not agree, not just because of the side formulas but also because the different closures in the two subproofs induce different essential expressions.) If one of these subderivations does not contain a side formula of the root inference (or generally a new typing introduced by the rule), we simply use that subderivation as our resulting T'. Otherwise, we weaken the two subderivations and construct the new proof. Explicitly, letting Σ_{1N} be $\Sigma_{2E} \backslash \Sigma_{1E}$, Σ_{2N} be $\Sigma_{1E} \backslash \Sigma_{2E}$, Γ_{1N} be $\Gamma_{2E} \backslash \Gamma_{1E}$ together with an occurrence of the principal expression of the root inference if necessary, and Γ_{2N} be $\Gamma_{1E} \backslash \Gamma_{2E}$ together with an occurrence of the principal expression of the root inference if necessary, we get:

$$\frac{\Sigma' \triangleright \Gamma', \mathbf{t}A \vee B^{\mu}_{X}}{\begin{array}{cc} \Sigma' \triangleright \Gamma' \mathbf{t}A \vee B^{\mu}_{X}, \mathbf{t}A^{\mu}_{X} & \Sigma' \triangleright \Gamma', \mathbf{t}A \vee B^{\mu}_{X}, \mathbf{t}B^{\mu}_{X} \\ T_1' + \Sigma_{1N} + \Gamma_{1N} & T_2 + \Sigma_{2N} + \Gamma_{2N} \end{array}}$$

Σ' is thus $\Sigma_{1E} \cup \Sigma_{2E}$, while Γ' contains the expression occurrences (other than the principal expression) either common to Γ_{1E} and Γ_{2E}, or present in Γ_{1N} or in Γ_{2N}. To show that the new tableau only contains essential inferences, it suffices to consider the new inference at the root. It is essential because one of the side expressions is essential. Likewise, we know that all elements of Σ' and Γ' are essential because each is essential in some subderivation. Finally, since the side expression tracks all the terms the principal expression tracks, the principal expression is essential. This shows that the constructed derivation has the required properties. The reasoning for the other cases is similar. ■

We are now ready to prove the main result.

Theorem 3 (Herbrand soundness) *Suppose Γ contains sentences of $L(CONST)$ (prefixed by ε). Then if there is a Herbrand proof of $\triangleright\Gamma$ then there is a closed first-order ground tableau for $\triangleright\Gamma$.*

Proof. Let T be a Herbrand proof of $\triangleright\Gamma$. By Lemma 20, we construct a Herbrand proof T' that respects the $<$ ordering on inferences. We then apply Lemma 22 to T' obtain a Herbrand proof T'' which respects $<$ and where a given Herbrand rule applies at most once on each branch. We then apply Lemma 23 to T'' to obtain a Herbrand proof T''' which respects $<$, where a given Herbrand rule applies at most once on each branch, and every inference is essential. At this point we can weaken T''' as necessary so that the end-sequent is again $\triangleright\Gamma$; call the result T^*. We will construct a closed first-order ground tableau by induction from T^*.

Place the first-order Hebrand terms h_i in one-to-one correspondence with first-order parameters a_i, and likewise place the modal Herbrand terms η_i in one-to-one correspondence with parameters α_i. If $\mathbf{u}e_X$ is a signed, tracked prefixed formula, let $\underline{\mathbf{u}e}$ denote the result of replacing each top-level first-order Herbrand term h_i in $\mathbf{u}e$

by a_i and each top-level modal Herbrand term η_i by α_i. By $\underline{\Gamma}$ denote the multiset consisting of $\underline{\mathbf{u}e}$ for each $\mathbf{u}e_X$ in Γ. Similarly, let $\underline{\mu/\mu\eta_i : i}$ and $\underline{t : \mu}$ be the results of replacing each top-level modal and first-order Herbrand term by the corresponding modal and first-order parameters, and extend the notation to $\underline{\Sigma}$.

We are given a Herbrand proof T^* whose root carries $\Sigma \triangleright \Gamma$, such that:

1. no Herbrand term that occurs in Γ is introduced by a Herbrand inference in T^*;

2. Σ is a Herbrand typing for Γ;

3. the side expression of any inference L in T^* is essential in some denominator;

4. no general inference L with instance x and no Herbrand rule H introducing x lies on a path from the root to a Herbrand rule introducing x.

We construct by induction a first-order ground proof of $\underline{\Sigma} \triangleright \underline{\Gamma}$. Since the proof T^* given by the application of these lemmas meets these conditions (insofar as Σ empty is a Herbrand typing for a sequent Γ containing no Herbrand terms), this construction will complete the proof of the theorem.

The base case is the closure rule; it suffices to observe that if Γ contains a complementary pair of literals so does $\underline{\Gamma}$.

Assuming the claim true for proofs of height n or less, consider a proof of height $n+1$. For boolean inferences, we apply the induction hypothesis to immediate subderivations and recombine, exploiting that $\underline{A \circ B} \equiv \underline{A} \circ \underline{B}$.

Suppose T^* ends in a general rule other than extra special necessity. We can take a positive necessity rule as a representative case. Then T^* looks thus:

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}\Box_i A^{\mu}_X}{\begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}\Box_i A^{\mu}_X, \mathbf{t}A^{\nu}_{X,N,\nu} \\ T^{\dagger}\end{array}}$$

We will apply the induction hypothesis to the immediate subderivation $T^{\dagger}$. We need to show first that no Herbrand term in the sequence N,ν is introduced by a Herbrand inference in $T^{\dagger}$. We know this because by assumption, in T^*, no general inference L with instance x lies on a path from the root to a Herbrand rule introducing x. We also need that Σ is a Herbrand typing for $\mathbf{t}A^{\nu}_{X,N,\nu}$. Now, by hypothesis the inference is essential; therefore some closure inference above has the form $\Sigma' \triangleright \Gamma'$ where $\Xi_{X,N,\nu} \subseteq \Sigma'$. In particular then Σ' contains a typing for each Herbrand term in the sequence N,ν. But we have seen that this typing cannot derive from an inference in $T^{\dagger}$. Therefore Σ also contains a typing for each Herbrand term in the sequence N,ν and since Σ is a Herbrand typing Σ too must contain $\Xi_{X,N,\nu}$. $T^{\dagger}$ inherits the remaining prerequisites from T^*.

We apply the induction hypothesis to obtain a closed first-order tableau T^o and apply the first-order necessity rule to obtain our result:

$$\dfrac{\underline{\Sigma} \triangleright \underline{\Gamma}, \underline{\mathbf{t}\Box_i A^{\mu}}}{\dfrac{\underline{\Sigma} \triangleright \underline{\Gamma}, \underline{\mathbf{t}\Box_i A^{\mu}}, \underline{\mathbf{t}A^{\nu}_{X,N,\nu}}}{T^o}}$$

Since we have established that $S, \Sigma \triangleright \mu/\nu : i$ we verify the needed side condition that $S, \underline{\Sigma} \triangleright \underline{\mu}/\underline{\nu} : i$.

Alternatively, suppose T^* ends in a Herbrand rule other than extra special necessity. We can take a positive possibility rule as a representative case. Then T^* looks thus:

$$\dfrac{\Sigma \triangleright \Gamma, \mathbf{t}\Diamond_i A^{\mu}_X}{\dfrac{\Sigma, \mu/\mu\eta : i \triangleright \Gamma, \mathbf{t}\Diamond_i A^{\mu}_X, \mathbf{t}A^{\mu\eta}_{X,\eta}}{T^{\dagger}}}$$

We will again apply the induction hypothesis to the immediate subderivation $T^{\dagger}$. Again, we show first that η is not introduced by a Herbrand inference in $T^{\dagger}$. We know this because by assumption, in T^*, no Herbrand inference L introducing x lies on a path from the root to another Herbrand rule introducing x. We also need that $\Sigma, \mu/\mu\eta : i$ is a Herbrand typing for $\mathbf{t}A^{\mu\eta}_{X,\eta}$. But η is the only new Hebrand term here and the new typing specifies an appropriate expression for it. So the induction hypothesis applies and the resulting proof T^o can be straightforwardly recombined with the appropriate rule to yield the desired result:

$$\dfrac{\underline{\Sigma} \triangleright \underline{\Gamma}, \underline{\mathbf{t}\Diamond_i A^{\mu}}}{\dfrac{\underline{\Sigma}, \underline{\mu/\mu\eta : i} \triangleright \underline{\Gamma}, \underline{\mathbf{t}\Diamond_i A^{\mu}}, \underline{\mathbf{t}A^{\mu\eta}}}{T^o}}$$

Finally, for the extra special necessity case, we simply combine the two pieces of reasoning to show that even with both the general and the Herbrand instantiation, the induction hypothesis extends to the subderivation and the first-order tableau rule can be reapplied to the result. ■

3.4 Completeness

The completeness proof for Herbrand proofs is straightforward by comparison; in fact, the ideas involved are implicit in the preceding discussion. We can simply rewrite a closed first-order tableau using the rules of the Herbrand tableau calculus.

Let $\Sigma \triangleright \Gamma$ be a first-order tableau line. Let σ_P be a map from the first-order parameters to first-order Herbrand terms, let σ_{κ} be a map from the modal parameters to modal Herbrand terms, and let σ_x be a function taking occurrences of formulas in Γ to sequences of Herbrand terms. Say $\sigma = \langle \sigma_P, \sigma_{\kappa}, \sigma_x \rangle$—we call σ a *Herbrandization*. For an expression $e = \mathbf{u}A^{\mu}$ in Γ we can introduce the notation $\sigma(\mathbf{u}A^{\mu})$ to denote $\mathbf{u}\sigma_P(A)^{\sigma_\kappa(\mu)}_{\sigma_x(e)}$ and indicate by $\sigma(\Gamma)$ the multiset $\{\sigma(e) \mid e \in \Gamma\}$. Likewise, we can define $\sigma(\Sigma)$ as:

$$\{\sigma_P(t) : \sigma_\kappa(\mu)(= \sigma(t : \mu)) \mid t : \mu \in \Sigma\} \cup \{\sigma_\kappa(\mu)/\sigma_\kappa(\nu) : i(= \sigma(\mu/\nu : i)) \mid \mu/\nu : i \in \Sigma\}$$

Theorem 4 (Completeness) *Let Γ be a set of formulas of $L(CONST)$ prefixed by ε. If there is a closed first-order ground tableau for $\triangleright\Gamma$ there is a Herbrand proof of $\triangleright\Gamma$.*

Proof. We prove by induction on the height of tableaux that we can construct a Herbrand proof whose root carries $\sigma(\Sigma) \triangleright \sigma(\Gamma)$ from a closed first-order ground tableau T whose root carries $\Sigma \triangleright \Gamma$ and a Herbrandization σ for which $\sigma(\Sigma)$ is a Herbrand typing for $\sigma(\Gamma)$. This will establish the result, because for the empty typing, $\sigma()$ remains the empty typing, which is a Herbrand typing for any multiset containing no first-order or modal parameters.

The base case is the closure rule; first we observe that if Γ contains a complementary pair of literals or the inconsistent literal so does $\sigma(\Gamma)$. Moreover, since $\sigma(\Sigma)$ is a Herbrand typing for $\sigma(\Gamma)$, certainly $\Xi_X \subseteq \sigma(\Sigma)$ for $X = \sigma_x(e)$ (for the requisite expression occurrences e). So the Herbrand closure rule applies to $\sigma(\Sigma) \triangleright \sigma(\Gamma)$.

Assuming the claim true for proofs of height n or less, consider a proof of height $n+1$. For boolean inferences, we apply the induction hypothesis to immediate subderivations and recombine, exploiting that $\sigma A \circ B \equiv \sigma A \circ \sigma B$.

Suppose T^* ends in a general rule other than extra special necessity. We can take a positive necessity rule as a representative case. Then T looks thus:

$$\cfrac{\Sigma \triangleright \Gamma, \mathbf{t}\Box_i A^\mu}{\begin{array}{c}\Sigma \triangleright \Gamma, \mathbf{t}\Box_i A^\mu, \mathbf{t}A^\nu \\ T'\end{array}}$$

From the side condition on instantiation in first-order tableaux, we know that $S, \Sigma \triangleright \mu/\nu : i$. Therefore $S, \sigma(\Sigma) \triangleright \sigma(\mu)/\sigma(\nu) : i$. It follows that there are terms $N, \sigma(\nu)$ that we can add to $X = \sigma_x(\mathbf{t}\Box_i A^\mu)$ such that $S, \Xi_{X,N,\nu} \triangleright \sigma(\mu)/\sigma(\nu) : i$. Hence we define σ'_x exactly like σ_x except $\sigma'_x(\mathbf{t}A^\nu) = X, N, \nu$. We apply the induction hypothesis to T' using $\sigma' = \langle \sigma_P, \sigma_\kappa, \sigma'_x \rangle$ and recombine using the Herbrand positive necessity rule.

Alternatively, suppose T ends in a Herbrand rule other than extra special necessity. We can take a positive possibility rule as a representative case. Then T looks thus:

$$\cfrac{\Sigma \triangleright \Gamma, \mathbf{t}\Diamond_i A^\mu}{\begin{array}{c}\Sigma, \mu/\mu\alpha : i \triangleright \Gamma, \mathbf{t}\Diamond_i A^\mu, \mathbf{t}A^{\mu\alpha} \\ T'\end{array}}$$

We will again apply the induction hypothesis to the immediate subderivation T' using a new Hebrandization σ'. To do this, we construct σ'_κ exactly like σ_κ except that if the principal expression is e, $\sigma'_\kappa(\alpha) = \eta = \eta_A(\sigma_x(e), \sigma_\kappa(\mu))$. We define σ'_x to be exactly like σ_x except $\sigma'_x(\mathbf{t}A^{\mu\alpha}) = \sigma_x(e), \eta$. We use $\sigma' = \langle \sigma_P, \sigma'_\kappa, \sigma'_x \rangle$. Since α does not occur in Γ or Σ, $\sigma'(\Sigma) = \sigma(\Sigma)$ and $\sigma'(\Gamma) = \sigma(\Gamma)$. Moreover, $\sigma'(\Sigma, \mu/\mu\alpha : i)$ must be a Herbrand typing for $\sigma'(\Gamma, e, \mathbf{t}A^{\mu\alpha})$ given the introduction of the appropriate Herbrand term η.

To obtain the needed Herbrand proof, we apply the positive Herbrand possibility rule to the resulting subderivation.

Finally, for the extra special necessity case, we simply combine the two pieces of reasoning to show that even with both the general and the Herbrand instantiation, the induction hypothesis extends to the subderivation and the first-order tableau rule can be reapplied to the result. ■

4 Lifted Deduction

The Herbrand tableau calculus affords flexible search for the *structure* of proofs, because of its permutabilities of inference. However, proof search in the Herbrand tableau calculus still suffers from nondeterminism at modal and quantifier rules, where a bound variable can be instantiated by an arbitrary concrete term. To describe computational proof search strategies precisely, we must find an alternative presentation of inference which lacks this nondeterminism. We will derive this calculus by the *lifting* constructions described in this section.

Lifting is a strategy that allows the choice of terms for instantiation at modal and quantifier rules to be delayed until sufficient information is available from the form of the proof to determine the value that is needed. Rules that require instantiation are reformulated to introduce a generic variable, called a *logic variable*, as a placeholder for the specific term which must ultimately be provided. Inference figures such as the closure rule which require terms to match introduce *constraints* on the values of logic variables. When inference figures have side conditions, the values of logic variables must be chosen in such a way that the side conditions are met; hence, side conditions are also reformulated to introduce appropriate constraints.

These various constraints are accumulated from a derivation in the lifted calculus. A constraint-satisfaction step is then required to obtain an ordinary derivation from the lifted derivation. In this step, we must construct a substitution of values to logic variables under which all of the constraints are satisfied. If we find such a substitution, we can apply the substitution to the lifted derivation to obtain a corresponding ordinary derivation. However, if there is no such substitution, no ordinary derivation corresponds to the lifted derivation.

In a lifted calculus, then, the nondeterminism of the choice of term for instantiation is factored from statement of the proof rules to the algorithm for constraint-satisfaction. But the constraint-satisfaction step, when analyzed in its own right, often turns out to have sharply delineated complexity. For example, in the case of first-order logic (without equality), a linear time *unification* algorithm suffices to solve the constraints associated with a deduction—see [Martelli and Montanari, 1982]. Hence the lifted calculus is useful not only for carrying out proof search in practice, but also, in many cases, for establishing theoretical bounds on the complexity of proof search problems in logical fragments [Lincoln and Shankar, 1994, Voronkov, 1996, McAllister and Rosenblitt, 1991].

This lifting strategy can be applied to any proof system; here, for example, we can use it for the ground proof system of section 2 or the Herbrand proof system of section 3. We treat the Herbrand proof system first in full detail, since this has the most direct relevance for practical proof search. Then we sketch how an analogous construction can be developed for the ground proof system. Our presentation of lifting in terms of constraint-satisfaction follows [Voronkov, 1996] most closely.

4.1 Formalism

We begin the construction of the lifted calculus by assuming a countable set X of logic variables disjoint from any of the symbols we have already considered. The ordinary logic will describe some set of terms T by which bound variables can be instantiated for the purposes of proof; in the lifted calculus we will simply use logic variables from X as placeholders. Thus, where the ordinary proof system deals with some function of expressions in a language parameterized by T—$\Phi(T)$—we set up a related set of expressions $\Phi(X)$ to work with in the lifted calculus.

Considering for example the Herbrand inference system, the ordinary terms consist of the two sets introduced in Definition 29: the set $\mathrm{CONST} \cup P_H$ describing constants and first-order Herbrand terms, and the set $\Pi(\kappa_\Upsilon)$ of Herbrand prefixes. It is convenient to partition X into countable sets X_P and X_κ to abstract these different kinds of terms. The basic expressions in Herbrand tableaux are prefixed formulas in the language $L(\mathrm{CONST} \cup P_H)^{\Pi(\kappa_\Upsilon)}$. Accordingly, lifted Herbrand tableaux appeal to prefixed formulas in the language $L(\mathrm{CONST} \cup X_P)^{X_\kappa}$; the formulas are then signed and tracked by variables in X:

Definition 36 (Parameterized signed, tracked expressions) *If E denotes the expressions of some class, then the* parameterized *signed, tracked expressions of that class are expressions of the form* $\mathbf{t}e_I$ *or* $\mathbf{f}e_I$ *where e is an expression of e and I is a finite sequence (possibly empty) of elements of* X*.*

The general correspondence between the logic variables in a proof, and the ordinary terms from T that the logic variables are meant to represent, is mediated by a *substitution*.

Definition 37 (Substitution) *A* substitution θ *is a partial mapping:* $X \to T$*, where* $\theta(x)$ *is defined for only finitely many variables x.*

A substitution can be represented as a finite set of ordered pairs $\theta = \{\langle x_1, t_1\rangle, \ldots \langle x_n, t_n\rangle\}$, where x_i are distinct variables and t_i are terms from T. For lifted Herbrand tableaux, we restrict our attention to substitutions which send variables in X_P to terms in $\mathrm{CONST} \cup P_H$ and which send variables in X_κ to prefixes in $\Pi(\kappa_\Upsilon)$.

Suppose φ is an expression in $\Phi(X)$ in which logic variables Z occur, and that θ is defined for all the logic variables in Z. By extension, we can apply θ to φ to give an expression $\theta(\varphi)$ in $\Phi(T)$ by replacing each occurrence of a logic variable x in φ with an occurrence of $\theta(x)$.

Substitutions may be subject to *constraints*; for a lifted derivation, a constraint expresses the conditions on a substitution that are required to obtain a corresponding ground proof. If a substitution θ meets the conditions provided by the constraint C, we say that θ satisfies C, written $\theta \models C$. The formulation of constraints depends on the conditions imposed by the tableau rules of a particular ordinary proof system; we now describe the constraints required for lifted Herbrand tableaux.

Let $T \subseteq X$ and let θ be a substitution defined on all the elements of T. We define a Herbrand typing $\Sigma_{\theta(T)}$ consisting of an appropriate expression $\mu/\mu\eta : i$ for each Herbrand prefix $\mu\eta \in \theta(T)$; and an appropriate expression $t : \mu$ for each first-order Herbrand term $t \in \theta(T)$. We continue to use $\Xi_{\theta(T)}$ to describe the Herbrand typing for Herbrand terms that *occur in* $\theta(T)$ (possibly as a subterm of a term in $\theta(T)$).

Definition 38 (Atomic Herbrand constraints) *The* atomic constraints *for lifted Herbrand tableaux take the following forms and impose the following conditions on substitutions:*

- *If X and T are lists of logic variables, then $\mathbf{D}(X;T)$ is an atomic constraint. $\theta \models \mathbf{D}(X;T)$ exactly when $\Xi_{\theta(X)} \subseteq \Sigma_{\theta(T)}$.*
- *If x and y are logic variables or constants, then $x = y$ is an atomic constraint. $\theta \models x = y$ exactly when $\theta(x) = \theta(y)$.*
- *If m and n are logic variables, A is a formula of $L(CONST \cup X_P)$ and X is a sequence of logic variables, then $\mathbf{P}_h(n, A, m; X)$ is an atomic constraint. $\theta \models \mathbf{P}_h(n, A, m; X)$ exactly when $\theta(n) = \theta(m)\,\eta_{\theta(A)}(\theta(m), \theta(X))$.*
- *If m and n are logic variables, i indexes a modality, and X is a sequence of logic variables, then $\mathbf{N}_h(m, n, i; X)$ is an atomic constraint. $\theta \models N(m, n, i; X)$ just in case there is a typing derivation $S, \Xi_{\theta(X)} \triangleright \theta(m)/\theta(n) : i$.*
- *If m and n are logic variables, A is a formula of $L(CONST \cup X_P)$, i indexes a modality and X is a sequence of logic variables, then $\mathbf{SN}_h(n, A, i, m; X)$ is an atomic constraint. $\theta \models \mathbf{SN}_h(n, A, i, m; X)$ just in case $\theta(n) = \theta(m)\,\eta_{s(i,\theta(A))}(\theta(m), \theta(X))$.*
- *If m, n and o are logic variables, A is a formula of $L(CONST \cup X_P)$, i indexes a modality and X is a sequence of logic variables, then $\mathbf{ESN}_h(o, A, i, m, n; X)$ is an atomic constraint. $\theta \models \mathbf{ESN}(o, A, i, m, n; X)$ just in case $\theta(o) = \theta(m)\,\eta_{e(i,\theta(m),\theta(A))}(\theta(n), \theta(X))$.*

- *If t and m are logic variables, A is a formula of $L(CONST \cup X_P)$, and X is a sequence of logic variables, then $\mathbf{E}_h(t, A, m; X)$ is an atomic constraint. $\theta \models \mathbf{E}_h(t, A, m; X)$ just in case $\theta(t) = h_{\theta(A)}(\theta(m), \theta(X))$.*

- *If t and m are logic variables and X is a set of logic variables, then $\mathbf{U}_h(t, m; X)$ is an atomic constraint. $\theta \models \mathbf{U}_h(t, m; X)$ just in case there is a typing derivation $S, \Xi_{\theta(X)} \triangleright \theta(t) : \theta(m)$.*

Constraints are assembled in proofs using two inductive constructions: conjunction and existential quantification:

Definition 39 (Herbrand constraints) *The* Herbrand constraints *include the atomic Herbrand constraints, together with constraints defined as follows:*

- *If C is a Herbrand constraint and D is a Herbrand constraint then $C \wedge D$ is a Herbrand constraint; $\theta \models C \wedge D$ just in case $\theta \models C$ and $\theta \models D$.*

- *If C is a Herbrand constraint and y is a logic variable, then $\exists y C$ is a constraint. $\theta \models \exists y C$ just in case there is some substitution θ' exactly like θ, except possibly in that θ' may assign a new value to y, such that $\theta' \models C$. If N is a finite sequence of logic variables $y_1, \ldots, y_n$, we will use $\exists N C$ to abbreviate $\exists y_1 \ldots \exists y_n C$.*

We can now describe the construction of lifted tableaux. In general, a line in a lifted tableau must pair a constraint with a specification from which an ordinary tableau line can be derived by substitution. For lifted Herbrand tableaux, we use as tableaux lines expressions of the form:

$$T \triangleright \Gamma \cdot C$$

T is a finite sequence of logic variables, Γ is a finite multiset of parameterized signed, tracked expressions of $L(\mathrm{CONST} \cup X_P)^{X_\kappa}$, and C is a Herbrand constraint. Our intention is that, on a substitution θ satisfying C, such an expression will correspond to the ordinary Herbrand tableau line $\Sigma_{\theta(T)} \triangleright \theta(\Gamma)$. We say a logic variable is *used* in a lifted tableau line if it occurs in T or if it occurs in Γ. In general, we will write tableau rules where any logic variable introduced in a denominator of a tableau rule cannot be used in the numerator. This is indicated by the parenthetical—v new—accompanying the specifications of the tableau rules.

Definition 40 (Tableau rule) *For lifted first-order multi-modal Herbrand deductions over a regime S, we will use the following tableau rules:*

1. *closure:*

$$\frac{T \triangleright \Gamma, \mathbf{t}R(s_1, \ldots, s_n)^{\mu}_{X}, \mathbf{f}R(t_1, \ldots, t_n)^{\nu}_{Y} \cdot \mathbf{D}(X; T) \wedge \mathbf{D}(Y; T) \wedge \mu = \nu \wedge s_1 = t_1 \wedge \ldots s_n = t_n}{\perp}$$

$$\frac{T \triangleright \Gamma, \mathbf{f}\top^{\mu}_{X} \cdot \mathbf{D}(X;T)}{\bot}$$

2. *conjunctive:*

$$\frac{T \triangleright \Gamma, \mathbf{t}A \wedge B^{\mu}_{X} \cdot C}{T \triangleright \Gamma, \mathbf{t}A \wedge B^{\mu}_{X}, \mathbf{t}A^{\mu}_{X}, \mathbf{t}B^{\mu}_{X} \cdot C} \qquad \frac{T \triangleright \Gamma, \mathbf{f}A \vee B^{\mu}_{X} \cdot C}{T \triangleright \Gamma, \mathbf{f}A \vee B^{\mu}_{X}, \mathbf{f}A^{\mu}_{X}, \mathbf{f}B^{\mu}_{X} \cdot C}$$

3. *disjunctive:*

$$\frac{T \triangleright \Gamma, \mathbf{f}A \wedge B^{\mu}_{X} \cdot C \wedge D}{T \triangleright \Gamma, \mathbf{f}A \wedge B^{\mu}_{X}, \mathbf{f}A^{\mu}_{X} \cdot C \qquad T \triangleright \Gamma, \mathbf{f}A \wedge B^{\mu}_{X}, \mathbf{f}B^{\mu}_{X} \cdot D}$$

$$\frac{T \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}_{X} \cdot C \wedge D}{T \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}_{X}, \mathbf{t}A^{\mu}_{X} \cdot C \qquad T \triangleright \Gamma, \mathbf{t}A \vee B^{\mu}_{X}, \mathbf{t}B^{\mu}_{X} \cdot D}$$

4. *negation:*

$$\frac{T \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X} \cdot C}{T \triangleright \Gamma, \mathbf{t}\neg A^{\mu}_{X}, \mathbf{f}A^{\mu}_{X} \cdot C} \qquad \frac{T \triangleright \Gamma, \mathbf{f}\neg A^{\mu}_{X} \cdot C}{T \triangleright \Gamma, \mathbf{f}\neg A^{\mu}_{X}, \mathbf{t}A^{\mu}_{X} \cdot C}$$

5. *possibility (n new):*

$$\frac{T \triangleright \Gamma, \mathbf{f}\Box_i A^{\mu}_{X} \cdot \exists n(C \wedge \mathbf{P}_h(n, \Box_i A, \mu; X))}{T, n \triangleright \Gamma, \mathbf{f}\Box_i A^{\mu}_{X}, \mathbf{f}A^{n}_{X,n} \cdot C}$$

$$\frac{T \triangleright \Gamma, \mathbf{t}\Diamond_i A^{\mu}_{X} \cdot \exists n(C \wedge \mathbf{P}_h(n, \Diamond_i A, \mu; X))}{T, n \triangleright \Gamma, \mathbf{t}\Diamond_i A^{\mu}_{X}, \mathbf{t}A^{n}_{X,n} \cdot C}$$

6. *necessity (n, N new):*

$$\frac{T \triangleright \Gamma, \mathbf{t}\Box_i A^{\mu}_{X} \cdot \exists n \exists N(C \wedge \mathbf{N}_h(\mu, n, i; X, N, n))}{T \triangleright \Gamma, \mathbf{t}\Box_i A^{\mu}_{X}, \mathbf{t}A^{n}_{X,N,n} \cdot C}$$

$$\frac{T \triangleright \Gamma, \mathbf{f}\Diamond_i A^{\mu}_{X} \cdot \exists n \exists N(C \wedge \mathbf{N}_h(\mu, n, i; X, N, n))}{T \triangleright \Gamma, \mathbf{f}\Diamond_i A^{\mu}_{X}, \mathbf{f}A^{n}_{X,N,n} \cdot C}$$

7. *special necessity—subject to the side condition that $\mathcal{A}(i)$ is one of KD, KDB, KD4, KD5 or KD45, that $i \leq j$ according to $\mathcal{N}$ (n new):*

$$\frac{T \triangleright \Gamma, \mathbf{t}\Box_j A^{\mu}_{X} \cdot \exists n(C \wedge \mathbf{SN}_h(n, \Box_j A, i, \mu; X))}{T, n \triangleright \Gamma, \mathbf{t}\Box_j A^{\mu}_{X}, \mathbf{t}A^{n}_{X,n} \cdot C}$$

$$\frac{T \triangleright \Gamma, \mathbf{f}\Diamond_j A^{\mu}_{X} \cdot \exists n(C \wedge \mathbf{SN}_h(n, \Box_j A, i, \mu; X))}{T, n \triangleright \Gamma, \mathbf{f}\Diamond_j A^{\mu}_{X}, \mathbf{f}A^{n}_{X,n} \cdot C}$$

8. *extra special necessity—subject to the side conditions that $A(i)$ is one of KD, KDB, KD4, KD5 or KD45, that $A(j)$ is one of K5, K45, KD5, KD45 or S5, that $i \leq j$ according to N (m, o, N new):*

$$\frac{T \triangleright \Gamma, \mathbf{u}A^{\nu}_{X} \cdot \exists m \exists o \exists N (C \wedge \mathbf{ESN}_h(o, A, i, m, \nu; X, N))}{T, o \triangleright \Gamma, \mathbf{t}\top^{o}_{X,N,o}, \mathbf{u}A^{\nu}_{X} \cdot C}$$

9. *existential (h new):*

$$\frac{T \triangleright \Gamma, \mathbf{t}\exists x A^{\mu}_{X} \cdot \exists h (C \wedge \mathbf{E}_h(h, \exists x A, \mu; X))}{T, h \triangleright \Gamma, \mathbf{t}\exists x A^{\mu}, \mathbf{t}A[h/x]^{\mu}_{X,h} \cdot C}$$

$$\frac{T \triangleright \Gamma, \mathbf{f}\forall x A^{\mu}_{X} \cdot \exists h (C \wedge \mathbf{E}_h(h, \forall x A, \mu; X))}{T, h \triangleright \Gamma, \mathbf{f}\forall x A^{\mu}_{X}, \mathbf{f}A[h/x]^{\mu}_{X,h} \cdot C}$$

10. *universal (z, N new):*

$$\frac{T \triangleright \Gamma, \mathbf{t}\forall x A^{\mu}_{X} \cdot \exists z \exists N (C \wedge \mathbf{U}_h(z, \mu; X, N, z))}{T \triangleright \Gamma, \mathbf{t}\forall x A^{\mu}_{X}, \mathbf{t}A[z/x]^{\mu}_{X,N,z} \cdot C}$$

$$\frac{T \triangleright \Gamma, \mathbf{f}\exists x A^{\mu}_{X} \cdot \exists z \exists N (C \wedge \mathbf{U}_h(z, \mu; X, N, z))}{T \triangleright \Gamma, \mathbf{f}\exists x A^{\mu}_{X}, \mathbf{f}A[z/x]^{\mu}_{X,N,z} \cdot C}$$

Once more, tableaux, branches, agreement and closure remain as in Definition 18 and following.

Definition 41 (Lifted Herbrand proof) *A* lifted Herbrand proof *consists of a closed, lifted Herbrand tableau for $\triangleright \Gamma \cdot C$ together with a substitution θ such that $\theta \models C$.*

The lifted calculus provides yet another sound and complete characterization of first-order modal models. To show this, we will prove the correspondence lifted Herbrand tableaux and ordinary Herbrand tableaux. In fact, we will set up a direct inductive correspondence between derivations in the two systems. This argument is presented in section 4.2.

4.2 Correctness

In this section, we prove the following theorem.

Theorem 5 (Correctness) *Let T be a (finite) sequence of logic variables, let Γ be a finite multiset of parameterized signed, tracked expressions of $L(CONST \cup X_P)^{X_\kappa}$, and let θ be a substitution defined on all logic variables that occur in T and Γ. Then the following conditions are equivalent:*

1. *There is a closed Herbrand tableau for* $\Sigma_{\theta(T)} \triangleright \theta(\Gamma)$.
2. *There is a constraint* C *and a closed lifted Herbrand tableau for* $T \triangleright \Gamma \cdot C$ *with* $\theta \models C$.

Proof (1 ⇒ 2). By induction on the structure of closed Herbrand tableaux.

The base case is a Herbrand tableau that consists of a closure inference; we take the binary closure as representative:

$$\frac{\Sigma_{\theta(T)} \triangleright \theta(\Gamma), \mathbf{t}\theta(A_1)^{\theta(\mu)}_{\theta(X)}, \mathbf{f}\theta(A_2)^{\theta(\nu)}_{\theta(Y)}}{\bot}$$

Now, since A_1 is an atomic formula, it takes the form $R(s_1,\ldots,s_n)$; and since A_2 is an atomic formula with $\theta(A_1) = \theta(A_2)$, A_2 takes the form $R(t_1,\ldots,t_n)$. Indeed, we must have $\theta(s_i) = \theta(t_i)$ for each i; since prefixes must match, we also have $\theta(\mu) = \theta(\nu)$. Moreover, because this is a Herbrand tableau, the side conditions $\Xi_{\theta(X)} \subseteq \Sigma_{\theta(T)}$ and $\Xi_{\theta(Y)} \subseteq \Sigma_{\theta(T)}$ hold. Therefore we have $\theta \models \mathbf{D}(X;T)$, $\theta \models \mathbf{D}(Y;T)$, and $\theta \models s_i = t_i$ for each i.

Thus it follows not just that the following is a closed lifted Herbrand tableau:

$$\frac{T \triangleright \Gamma, \mathbf{t}R(s_1,\ldots,s_n)^{\mu}_{X}, \mathbf{f}R(t_1,\ldots,t_n)^{\nu}_{Y} \cdot \mathbf{D}(X;T) \wedge \mathbf{D}(Y;T) \wedge \mu = \nu \wedge s_1 = t_1 \wedge \ldots s_n = t_n}{\bot}$$

but also that θ satisfies the associated constraint.

Now, assuming the hypothesis holds for closed Herbrand tableaux of height h or less, consider a closed Herbrand tableau $\mathcal{T}$ of height $h+1$. We construct the corresponding closed lifted Herbrand tableau by case analysis on the inference at the root of $\mathcal{T}$.

As a representative Boolean inference (the other cases are similar), we consider $\mathcal{T}$ of the form

$$\frac{\Sigma_{\theta(T)} \triangleright \theta(\Gamma), \mathbf{f}\theta(A) \wedge \theta(B)^{\theta(\mu)}_{\theta(X)}}{\begin{array}{c}\Sigma_{\theta(T)} \triangleright \theta(\Gamma), \mathbf{f}\theta(A) \wedge \theta(B)^{\theta(\mu)}_{\theta(X)}, \mathbf{f}\theta(A)^{\theta(\mu)}_{\theta(X)} \\ \mathcal{T}'\end{array} \qquad \begin{array}{c}\Sigma_{\theta(T)} \triangleright \theta(\Gamma), \mathbf{f}\theta(A) \wedge \theta(B)^{\theta(\mu)}_{\theta(X)}, \mathbf{f}\theta(B)^{\theta(\mu)}_{\theta(X)} \\ \mathcal{T}''\end{array}}$$

This analysis of $\mathcal{T}$ exploits the fact that $\theta(A \wedge B)$ is $\theta(A) \wedge \theta(B)$. Application of the induction hypothesis to $\mathcal{T}'$ and $\mathcal{T}''$ yields closed lifted Herbrand tableaux $\mathcal{T}'_L$ and $\mathcal{T}''_L$:

$$\begin{array}{c}T \triangleright \Gamma, \mathbf{f}A \wedge B^{\mu}_{X}, \mathbf{f}A^{\mu}_{X} \cdot C \\ \mathcal{T}'_L\end{array} \qquad \begin{array}{c}T \triangleright \Gamma, \mathbf{f}A \wedge B^{\mu}_{X}, \mathbf{f}B^{\mu}_{X} \cdot D \\ \mathcal{T}''_L\end{array}$$

It also yields that $\theta \models C$ and $\theta \models D$. Thus we can recombine $\mathcal{T}'_L$ and $\mathcal{T}''_L$ by the same inference figure, to obtain the needed closed lifted Herbrand tableau (schematized below) with a constraint $C \wedge D$ that θ satisfies:

$$\frac{T \triangleright \Gamma, \mathbf{f}A \wedge B^{\mu}_{X} \cdot C \wedge D}{T'_L \qquad\qquad T''_L}$$

Suppose T ends in a possibility inference, for (a representative) example thus:

$$\cfrac{\Sigma_{\theta(T)} \triangleright \theta(\Gamma), \mathbf{t}\Diamond_i \theta(A)^{\theta(\mu)}_{\theta(X)}}{\begin{array}{c}\Sigma_{\theta(T)}, \theta(\mu)/\theta(\mu)\eta \triangleright \theta(\Gamma), \mathbf{t}\Diamond_i \theta(A)^{\theta(\mu)}_{\theta(X)}, \mathbf{t}\theta(A)^{\theta(\mu)\eta}_{\theta(X),\theta(\mu)\eta} \\ T'\end{array}}$$

Let us introduce a fresh variable n for which θ is undefined (thereby working to satisfy the lifted tableau novelty condition). Then we can define θ' which agrees with θ everywhere θ is defined, and where moreover $\theta'(n) = \theta(\mu)\eta$. Under these conditions $\theta' \models \mathbf{P}_h(n, \Diamond_i A, \mu; X))$, reflecting the side condition on the choice of η for the (ordinary) Herbrand possibility inference.

Now the root of T' carries a tableau line which may be written

$$\Sigma_{\theta'(T),\theta'(n)} \triangleright \theta'(\Gamma), \mathbf{t}\Diamond_i \theta'(A)^{\theta'(\mu)}_{\theta'(X)}, \mathbf{t}\theta'(A)^{\theta'(n)}_{\theta'(X),\theta'(n)}$$

We can therefore apply the induction hypothesis to obtain a closed lifted Herbrand tableau T'_L, and construct:

$$\cfrac{T \triangleright \Gamma \mathbf{t}\Diamond_i A^{\mu}_{X} \cdot \exists n(C \wedge \mathbf{P}_h(n, \Diamond_i A, \mu; X))}{\begin{array}{c}T, n \triangleright \Gamma, \mathbf{t}\Diamond_i A^{\mu}_{X}, \mathbf{t}A^{n}_{X,n} \cdot C \\ T'_L\end{array}}$$

Now by induction $\theta' \models C$. It follows from the definition of θ' and our earlier observation that $\theta \models \exists n(C \wedge \mathbf{P}_h(n, \Diamond_i A, \mu; X))$—hence the constructed derivation suffices. Special necessity and existential inferences require similar reasoning—the introduction of Herbrand terms for special necessity and existential inferences allow the application for those inference figures of the strategy for proof transformation illustrated here for the possiblity inference.

Next, suppose T ends in a necessity inference, for (a representative) example thus:

$$\cfrac{\Sigma_{\theta(T)} \triangleright \theta(\Gamma), \mathbf{t}\Box_i \theta(A)^{\theta(\mu)}_{\theta(X)}}{\begin{array}{c}\Sigma_{\theta(T)} \triangleright \theta(\Gamma), \mathbf{t}\Box_i \theta(A)^{\theta(\mu)}_{\theta(X)}, \mathbf{t}\theta(A)^{\nu}_{\theta(X),L,\nu}, \\ T'\end{array}}$$

We introduce a fresh variable n_0 to correspond to ν and a variable n_i for each term l_i in L (thereby working to satisfy the lifted tableau novelty condition); by N denote the sequence $n_1, \ldots, n_k$. We introduce a substitution θ' which extends θ such that $\theta(n_0) = \nu$ and such that $\theta(n_i) = l_i$; thus $\theta(N) = L$. Again, we now have $\theta' \models \mathbf{N}_h(\mu, n, i; X, N, n)$, because of the side condition on the (ordinary) Herbrand necessity inference.

Here the root of T' carries a tableau line which may be written

$$\Sigma_{\theta'(T)} \triangleright \theta'(\Gamma), \mathbf{t}\Box_i \theta'(A)^{\theta'(\mu)}_{\theta'(X)}, \mathbf{t}\theta'(A)^{\theta'(n)}_{\theta'(X),\theta'(N),\theta'(n)}$$

We can therefore apply the induction hypothesis to T' to obtain a closed lifted Herbrand tableau T'_L, and construct:

$$\cfrac{T \triangleright \Gamma \mathbf{t}\Box_i A^{\mu}_{X} \cdot \exists n \exists N (C \wedge \mathbf{N}_h(\mu, n, i; X, N, n))}{\begin{array}{c} T \triangleright \Gamma, \mathbf{t}\Box_i A^{\mu}_{X}, \mathbf{t}A^{n}_{X,N,n} \cdot C \\ T'_L \end{array}}$$

Again by induction $\theta' \models C$, and hence $\theta \models \exists n \exists N (C \wedge \mathbf{N}_h(\mu, n, i; X, N, n))$. Universal inferences can be handled by similar reasoning; indeed, despite the double condition involved, so can extra special necessity inferences.

Proof (2 ⇒ 1). By induction on the structure of closed *lifted* Herbrand tableaux.

The base case is a lifted Herbrand tableau that consists of a closure inference; again, we take the binary closure as representative:

$$\cfrac{T \triangleright \Gamma, \mathbf{t}R(s_1, \ldots, s_n)^{\mu}_{X}, \mathbf{f}R(t_1, \ldots, t_n)^{\nu}_{Y} \cdot \mathbf{D}(X;T) \wedge \mathbf{D}(Y;T) \wedge \mu = \nu \wedge s_1 = t_1 \wedge \ldots s_n = t_n}{\perp}$$

By assumption, we have a substitution θ with

$$\theta \models \mathbf{D}(X;T) \wedge \mathbf{D}(Y;T) \wedge \mu = \nu \wedge s_1 = t_1 \wedge \ldots s_n = t_n$$

Now, we construct

$$\cfrac{\Sigma_{\theta(T)} \triangleright \theta(\Gamma), \mathbf{t}R(\theta(s_1), \ldots, \theta(s_n))^{\theta(\mu)}_{\theta(X)}, \mathbf{f}R(\theta(t_1), \ldots, \theta(t_n))^{\theta(\nu)}_{\theta(Y)}}{\perp}$$

We must show that this has the form required for an ordinary Herbrand closure inference. The satisfaction of the constraint guarantees this. In particular, we have $R(\theta(s_1), \ldots, \theta(s_n))^{\theta(\mu)} = R(\theta(t_1), \ldots, \theta(t_n))^{\theta(\nu)}$, thanks to the equalities which θ satisfies. Moreover, we have $\Xi_{\theta(X)} \subseteq \Sigma_{\theta(T)}$ and $\Xi_{\theta(Y)} \subseteq \Sigma_{\theta(T)}$ in virtue of the constraints $\mathbf{D}(X;T)$ and $\mathbf{D}(Y;T)$ that θ satisfies.

Now, assuming the hypothesis holds for closed lifted Herbrand tableaux of height h or less, consider a closed lifted Herbrand tableau T of height $h+1$. We construct the corresponding closed ordinary Herbrand tableau by case analysis on the inference at the root of T. Again, the flavor for the reasoning required in each case is provided by the example cases of a boolean inference, a possibility inference and a necessity inference.

First, then, consider as T:

$$\frac{T \triangleright \Gamma, \mathbf{f}A \wedge B^{\mu}_{X} \cdot C \wedge D}{\begin{array}{cc} T \triangleright \Gamma, \mathbf{f}A \wedge B^{\mu}_{X}, \mathbf{f}A^{\mu}_{X} \cdot C & T \triangleright \Gamma, \mathbf{f}A \wedge B^{\mu}_{X}, \mathbf{f}B^{\mu}_{X} \cdot D \\ T' & T'' \end{array}}$$

By assumption, we have a substitution θ such that $\theta \models C \wedge D$. Hence $\theta \models C$ and $\theta \models D$; thus the induction hypothesis applies to T' and T''. We combine the resulting derivations T'_O and T''_O into the following closed ordinary Herbrand tableau, as needed:

$$\frac{\Sigma_{\theta(T)} \triangleright \theta(\Gamma), \mathbf{f}\theta(A) \wedge \theta(B)^{\theta(\mu)}_{\theta(X)}}{\begin{array}{cc} \Sigma_{\theta(T)} \triangleright \theta(\Gamma), \mathbf{f}\theta(A) \wedge \theta(B)^{\theta(\mu)}_{\theta(X)}, \mathbf{f}\theta(A)^{\theta(\mu)}_{\theta(X)} & \Sigma_{\theta(T)} \triangleright \theta(\Gamma), \mathbf{f}\theta(A) \wedge \theta(B)^{\theta(\mu)}_{\theta(X)}, \mathbf{f}\theta(B)^{\theta(\mu)}_{\theta(X)} \\ T'_O & T''_O \end{array}}$$

We exploit again the fact that $\theta(A \wedge B)$ is $\theta(A) \wedge \theta(B)$.

Next, consider T constructed by a possibility inference:

$$\frac{T \triangleright \Gamma, \mathbf{t}\Diamond_i A^{\mu}_{X} \cdot \exists n(C \wedge \mathbf{P}_h(n, \Diamond_i A, \mu; X))}{\begin{array}{c} T, n \triangleright \Gamma, \mathbf{t}\Diamond_i A^{\mu}_{X}, \mathbf{t}A^{n}_{X,n} \cdot C \\ T' \end{array}}$$

By assumption we have $\theta \models \exists n(C \wedge \mathbf{P}_h(n, \Diamond_i A, \mu; X))$; therefore there is some θ' that differs from θ only in n such that $\theta' \models C$ and $\theta' \models \mathbf{P}_h(n, \Diamond_i A, \mu; X)$. Moreover, since n is not used in the root tableau line, $\theta(\Gamma) = \theta'(\Gamma)$ and $\theta(T) = \theta'(T)$.

Apply the induction hypothesis to T' and θ', to obtain T'_O; I claim that this permits the construction of a closed ordinary Herbrand tableau thus:

$$\frac{\Sigma_{\theta(T)} \triangleright \theta(\Gamma), \mathbf{t}\Diamond_i \theta(A)^{\theta(\mu)}_{\theta(X)}}{\begin{array}{c} \Sigma_{\theta(T)}, \theta(\mu)/\theta(\mu)\,\eta : i \triangleright \theta(\Gamma), \mathbf{t}\Diamond_i \theta(A)^{\theta(\mu)}_{\theta(X)}, \mathbf{t}\theta(A)^{\theta(\mu)\,\eta}_{\theta(X), \theta(\mu)\eta} \\ T'_O \end{array}}$$

It suffices to justify the analysis of the root of T'_O. Start with the typing: the induction hypothesis gives $\Sigma_{\theta'(T), \theta'(n)}$, which is $\Sigma_{\theta(T)}, \Sigma_{\theta'(n)}$. Now since $\theta' \models \mathbf{P}_h(n, \Diamond_i A, \mu; X)$, $\theta'(n) = \theta'(\mu)\,\eta$ where $\eta = \eta_{\theta'(A)}(\theta'(\mu), \theta'(X))$. Again since n is new that means $\theta'(n) = \theta(\mu)\,\eta_{\theta(A)}(\theta(\mu), \theta(X))$, and hence the added typing is indeed $\theta(\mu)/\theta(\mu)\,\eta : i$. Likewise, the remainder of the tableau line is justified because $\theta'(n) = \theta(\mu)\,\eta$ and, with n new, θ agrees elsewhere with θ'.

Finally, consider T constructed by a necessity inference:

$$\frac{T \rhd \Gamma, \mathbf{t}\Box_i A^{\mu}_{X} \cdot \exists n \exists N (C \wedge \mathbf{N}_h(\mu, n, i; X, N, n))}{\begin{array}{c} T \rhd \Gamma, \mathbf{t}\Box_i A^{\mu}_{X}, \mathbf{t}A^{n}_{X,N,n} \cdot C \\ T' \end{array}}$$

Again, by assumption, we have $\theta \models \exists n \exists N (C \wedge \mathbf{N}_h(\mu, n, i; X, N, n))$; therefore there is some θ' that differs from θ only in new n and N such that θ' models C and $\theta' \models \mathbf{N}_h(\mu, n, i; X, N, n)$. Use $\theta'(n) = \nu$ and $\theta'(N) = L$. Again, we apply the induction hypothesis to T' and θ', to obtain T'_O and construct a closed ordinary Herbrand tableau:

$$\frac{\Sigma_{\theta(T)} \rhd \theta(\Gamma), \mathbf{t}\Box_i \theta(A)^{\theta(\mu)}_{\theta(X)}}{\begin{array}{c} \Sigma_{\theta(T)} \rhd \theta(\Gamma), \mathbf{t}\Box_i \theta(A)^{\theta(\mu)}_{\theta(X)}, \mathbf{t}\theta(A)^{\nu}_{\theta(X),L,\nu} \\ T'_O \end{array}}$$

Here we rewrite $\Sigma_{\theta'(T)}$ as $\Sigma_{\theta(T)}$ and $\theta'(\Gamma)$ as $\theta(\Gamma)$ straightforwardly since θ' and θ agree on variables used in the root. So we require only that there is a typing derivation $S, \Xi_{\theta(X),L,\nu} \rhd \theta(\mu)/\nu : i$. This follows from the fact that $\theta' \models \mathbf{N}_h(\mu, n, i; X, N, n)$. ■

4.3 Discussion and Extensions

The argument of Theorem 5 in fact shows that a lifted proof and a corresponding ordinary proof consist of corresponding inferences applied in the same order. Here we will suggest informally some consequences of this property. In [Voronkov, 1996], the property is formalized, using the notion of a *skeleton* of a derivation; the skeleton of a derivation is a tree that encodes the identity of inferences performed but abstracts out from instantiations made at quantifier (or modal) inferences. With this abstraction, Voronkov shows that the lifted proof and the ordinary proof put in correspondence in his correctness theorem share the same skeleton; the result would carry over straightforwardly in the case of Theorem 5.

Now, in section 3, we introduced Herbrand tableaux over ground tableaux in order to eliminate the impermutability of inference associated with eigenvariable conditions on ground quantifier rules, which require the use of new parameters. Then, in *lifting* Herbrand tableaux, we reintroduced similar conditions by requiring the use of new logic variables in the lifted quantifier rules. Nevertheless, from the correspondence between lifted tableaux and Herbrand tableaux, the lifted tableaux must retain free permutabilities of inference. That is, the argument of Theorem 5 can be used to accomplish the interchange of any two inferences in a lifted tableau, by first finding the corresponding ordinary Herbrand tableau, interchanging the corresponding inferences in the Herbrand tableau, and rederiving a corresponding lifted tableau.

The reason lifted tableaux retain free interchange of inferences despite the novelty condition on logic variables is because *all* instantiation inferences in lifted tableaux involve fresh logic variables. In the ground calculus, inferences with universal force may be locked into a constrained position in a tableau because of the term selected for instantiation; there are no figures in lifted tableaux that range over logic variables with a corresponding universal force and hence no inferences to be locked in by the novelty condition on logic variables.

In fact, we can generally rewrite lifted Herbrand tableaux to strengthen the novelty of logic variables, without affecting provability; this result allows constraints to be simplified for algorithmic purposes.

Definition 42 (Pure variable tableaux) *An inference in a lifted tableau is a* pure variable inference *if each logic variable introduced at that rule is* used *only in the subtableau rooted at that inference. (Recall x is* used *in* $T \triangleright \Gamma \cdot C$ *only if x occurs in T or in* Γ*.) A* pure variable lifted tableau *is one in which every inference is a pure variable inference.*

The pure variable property strengthens the novelty condition on logic variables from a local property of the sequent at which an inference applies to a global property of the tableau in which the inference applies. Yet any closed lifted Herbrand tableau whose root carries $T \triangleright \Gamma \cdot C$ can be transformed into a pure-variable closed lifted Herbrand tableau whose root carries $T \triangleright \Gamma \cdot C'$ where $\theta \models C$ just in case $\theta \models C'$. To accomplish this transformation, it suffices to rename variables appropriately throughout the proof; the argument is a straightforward application of arguments that yield pure variable proofs for sequent systems for classical logic—see for example [Gallier, 1986, pp 274–276]. Thus the constraint C' that results is an alphabetic variant of C, but each quantifier $\exists x$ in C' binds a distinct variable x.

To see the algorithmic simplification this affords, consider a pure-variable closed lifted Herbrand tableau T whose root carries $\triangleright \Gamma \cdot C$, where Γ is a multiset of formulas from $L(\text{CONST})$ labeled with the empty prefix. The conditions on T are characteristic of problems of modal deduction; that is, the problem statement is formulated without recourse to terms used for the purposes of proof. In particular, no logic variables occur in Γ; $\theta(\Gamma) = \Gamma$ for any substitution.

Meanwhile, with these conditions on T, it is straightforward to reformulate C into the form $C' = \exists VA$ where A is a conjunction of atomic constraints, with $\theta \models C$ just in case $\theta \models C'$. We simply lift the existential quantifiers in C (each of which binds a distinct variable) into prenex position. We can therefore conclude (thanks to Theorems 3 and 5) that there is a closed ground tableau $\triangleright \Gamma$ just in case there is a substitution θ satisfying each of the atomic constraints in A.

We thus arrive at a general perspective on the selection of terms in lifted deduction as constructing a substitution to satisfy certain equalities and certain other primitive constraints (governing the types of values for variables and the occurrence of values of variables as subterms of other terms). This perspective is assumed in algorithmic characterizations of instantiation, both for classical inference (e.g., in work on unification [Martelli and Montanari, 1982]) and for equational or constrained reasoning in modal inference [Wallen, 1990, Frisch and Scherl, 1991, Auffray and Enjalbert, 1992, Ohlbach, 1993, Otten and Kreitz, 1996, Schmidt, 1998, Stone, 1999c].

The opening of sections 4 and 4.1 indicated that the lifting construction applies in a similar manner across proof systems. It is thus no surprise that the results outlined here for lifted modal Herbrand deduction mesh are of a piece with many related systems. To underscore the point, we close this section by sketching a parallel lifting construction for ground first-order modal tableaux with parameters and eigenvariable conditions. Lifted ground tableaux will not be appropriate for general modal inference; as we shall see, they (like any lifted system) retain the permutabilities and impermutabilities of ordinary ground tableaux. However, they may provide a useful technique for deduction in logical fragments where the impermutabilities are not an issue (such as the simple fragments that are "interpreted" by proof search in the design of logic programming languages).

The lifting construction again begins with the definition of constraints, parallel to the definition of constraints in Definition 38. For the ground calculus, these must be designed to enforce any side conditions from the ground inference figures of Definition 17. For example, we need these constraints for possibility and necessity inferences:

Definition 43 (Atomic constraints) *The* atomic constraints *for lifted ground tableaux include:*

- *If n and m are logic variables and T is a sequence of logic variables, then $\mathbf{P}_g(n,m;T)$ is an atomic constraint. $\theta \models \mathbf{P}_g(n,m;T)$ just in case $\theta(n)$ takes the form $\theta(m)\,\alpha$ for some modal parameter α and α does not occur in $\theta(T)$.*

- *If n and m are logic variables, Σ is a typing (containing logic variables), and i indexes a modality, then $\mathbf{N}_g(n,m,i;\Sigma)$ is an atomic constraint. $\theta \models \mathbf{N}_g(n,m,i;\Sigma)$ just in case there is a typing derivation $S, \theta(\Sigma) \triangleright \theta(m)/\theta(n) : i$.*

Of course, the general constraints of equality, conjunction and existential quantifications continue to be needed.

The lifting construction continues by the adaptation of tableau rules to manipulate constrained sequents; these will take the form $\Sigma \triangleright \Gamma \cdot C$. (Since Σ will abstract a typing, we will want to use V_Σ to designate the logic variables that occur in Σ.) As before, boolean rules generally decompose the principal formula in the numerator of the tableau rule while conjoining constraints from all the denominators of the rules. Now the rules that require specific instantiations in the ground calculus require special reformulation; each revised inference introduces some new logic variables, and (using an existential quantifier) constrains these variables appropriately to match any side conditions on the application of the ordinary rule. Again, we limit ourselves to examples:

Definition 44 (Lifted Tableau Rule) *The lifted version of the ground modal deductions (over a regime S) include:*

1. *possibility (n new):*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}\Diamond_i A^{\mu} \cdot \exists n(C \wedge \mathbf{P}_g(n,\mu;V_{\Sigma}))}{\Sigma, m/n : i \triangleright \Gamma, \mathbf{t}\Diamond_i A^{\mu}, \mathbf{t}A^n \cdot C}$$

2. *necessity (n new):*

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}\Box_i A^{\mu} \cdot \exists n(C \wedge \mathbf{N}_g(n,m,i;\Sigma))}{\Sigma \triangleright \Gamma, \mathbf{t}\Box_i A^{\mu}, \mathbf{t}A^n \cdot C}$$

Finally, we establish a correctness theorem:

Theorem 6 (Correctness) *Let Σ be a typing containing variables, let Γ be a finite multiset of parameterized signed expressions (in an appropriate language), and let θ be a substitution defined on all logic variables that occur in Σ and Γ, such that $\theta(\Sigma)$ is a typing for $\theta(\Gamma)$. Then the following conditions are equivalent:*

1. *There is a closed ground tableau for $\theta(\Sigma) \triangleright \theta(\Gamma)$.*

2. *There is a constraint C and a lifted tableau for $\Sigma \triangleright \Gamma \cdot C$ with $\theta \models C$.*

The **proof** again consists of an induction on the structure of ordinary tableaux and an induction on the structure of lifted tableaux. Indeed, the arguments can be essentially preserved from the proof of Theorem 5. For $(1 \Rightarrow 2)$, the form of the ordinary tableau, in meeting any side conditions on the inferences, guarantees that a secondary substitution can be constructed from θ so as to allow the induction hypothesis to be applied and the resulting derivation(s) to be (re)assembled into an overall lifted tableau, with the resulting overall constraint satisfied by θ. For $(2 \Rightarrow 1)$, the fact that θ meets the constraint and the use of fresh variables ensures that the the induction hypothesis can be applied and that the resulting derivations(s) can be (re)assembled into an overall ground tableau. ■

We can observe from this proof strategy that since ground proofs do not enjoy free interchange of inference, lifted proofs cannot enjoy free interchange of inference either. It is of course possible in this case to understand these impermutabilities directly from the lifted calculus itself (much as we did earlier in this section). In this case, the lifted rules are not permutable because the constraint derived for a lifted tableau T need not be equivalent to the constraint derived from a tableau resulting from the interchange of inferences in T. So one constraint may not be satisfiable while the other is; hence one lifted tableau may not represent a lifted proof while the other does.

5 Specialized Systems

Thus far, we have considered just the general problem of inference in first-order multi-modal logic. We have allowed for an arbitrary number of modalities, subject to accessibility relations that could be any of serial, reflexive, transitive, euclidean, or narrowing (to another modality); we have treated a full complement of boolean, first-order and modal connectives.

In knowledge representation, domains may invite a more constrained inventory of modal operators or connectives—for example by motivating an S4 model of agents' knowledge rather than an S5 model (or vice versa), or by suggesting axiomatic theories that are naturally formulated as modal Horn clauses (or some other logical fragment). The results we have presented are compatible with such restrictions. Because the results are parameterized by the modal regime, they can be applied for simpler theories of modalities, right down to the simplest modal logic of a single K operator. And because all the results are formulated without reference to a cut inference rule, inference calculi for logical fragments can be obtained simply by omitting the inference figures for connectives excluded from the fragment.

However, while some complexities of the inference system—take the special necessity inference of ordinary Herbrand tableaux as an example—come clearly linked with the complexities of the modal regime and the logical language which motivate their use, not all complexities of the inference system do. This section concentrates on two such features of ordinary Herbrand tableaux: the side conditions on typings for closure inferences, and the introduction of auxiliary tracking terms as a side-effect of instantiation. Both of these are indirect reflections of the expressive power of our general framework for first-order multi-modal logic, and can be eliminated for a range of simpler deduction tasks.

The justification of the simplifications derive from elaborations of the Herbrand soundness theorem, Theorem 3. Recall that the theorem is proved by first transforming any closed Herbrand tableau T into a revised closed Herbrand tableau T^* by interchange and omission of inferences. In this T^*, the natural ordering $<$ on inferences is respected, a given Herbrand rule applies at most once on each branch, and every inference is essential. An induction—the chief obligation of which is to ensure that the local typing conditions imposed by the ground inference figures are met—converts T^* to a ground proof. Thus, the simplifications we consider here provide ways of establishing additional properties of the revised proof T^*, and thereby rewriting T^* using ground inference figures by alternative means.

5.1 Terminology and a basic lemma

To accomplish these transformations, we will be working with *relaxed* Herbrand tableaux. The *relaxed Herbrand closure rules* are given by the figures

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}A^{\mu}_{X}, \mathbf{f}A^{\mu}_{Y}}{\bot} \qquad \frac{\Sigma \triangleright \Gamma, \mathbf{f}\top^{\mu}_{X}}{\bot}$$

for A an atomic formula. A *relaxed Herbrand tableau* is a tableau built in accordance with the relaxed Herbrand closure rules and the recursive tableau rules of Definition 32.

Suppose $\Phi(\Sigma, E)$ is a property of Herbrand typings and signed, tracked prefixed formulas (in a language with Herbrand terms). Then we say a relaxed Herbrand tableau T *satisfies* Φ just in case for each binary closure inference in T:

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}A^{\mu}_{X}, \mathbf{f}A^{\mu}_{Y}}{\bot}$$

both $\Phi(\Sigma, \mathbf{t}A^{\mu}_{X})$ and $\Phi(\Sigma, \mathbf{f}A^{\mu}_{Y})$ hold, and for each unary closure inference in T:

$$\frac{\Sigma \triangleright \Gamma, \mathbf{f}\top^{\mu}_{X}}{\bot}$$

$\Phi(\Sigma, \mathbf{f}\top^{\mu}_{X})$ holds. For example, consider $\Phi(\Sigma, \mathbf{u}A^{\mu}_{X})$ defined as $\Xi_X \subseteq \Sigma$. Then a relaxed Herbrand tableau satisfying Φ is in fact a Herbrand tableau. One can therefore imagine transforming a relaxed Herbrand tableau so as to satisfy increasingly strong Φ, until what we have is in fact a Herbrand tableau. In essence, that is exactly what we will do.

Any defect of a relaxed Herbrand tableau comes in how symbols are introduced in the proof. Let us say that occurrence of a symbol t in a term is *unchecked* in a closure inference with typing Σ if t occurs in a principal expression of the inference and $\Xi_t \not\subseteq \Sigma$. Then a relaxed Herbrand tableaux fails to be a Herbrand tableau in virtue of its unchecked symbol occurrences.

The transformations that remedy such defects consequently require us to replace one symbol for another. For example, we might have a case in a relaxed Herbrand tableau where a first-order Herbrand term h which is not properly introduced is instantiated for x in a general inference (for example to reason with a universal formula $\forall x A$). Then occurrences of h may be unchecked elsewhere in the tableau. But informally, because h is not introduced by a rule which precisely requires it, we should be able to replace h with another symbol whose corresponding occurrences will not be unchecked, such as a constant c. Likewise, we might have a case in a relaxed Herbrand tableau where a modal Herbrand term η which is not properly introduced appears in the transition taken in a general modal inference (say to a necessary formula $\Box_i A$); occurrences of η may be unchecked elsewhere in the tableau. In this case we might rewrite the inference to use a special necessity inference and hence introduce a different modal Herbrand term η' whose corresponding occurrences will not be unchecked.

Our reasoning about relaxed Herbrand tableaux therefore require results showing that we can systematically vary the choice of certain kinds of symbols in the proof. Such results resemble the pure variable result of the previous section.

However, with relaxed Herbrand tableaux the replacement involves repercussions throughout the proof in the structure of Herbrand terms; it is not surprising that the such changes are possible, but the exact transformation required takes some technical effort to spell out precisely.

Accordingly, we need some further definitions. Say that a Herbrand term or Herbrand prefix p is *implicated* in an expression E if p occurs in E or if some Herbrand function f_A depending on a formula A occurs in E and p is implicated in A. Say q is independent of p if p is not implicated in q. Say q is *type-compatible* with p if whenever there is a derivation of $S, \Xi_X \triangleright J$ then there is a derivation of $S, \Xi_{X[q/p]} \triangleright J[q/p]$.

Definition 45 (renaming) *Let σ be a map taking finitely many Herbrand terms h into a corresponding Herbrand term $\sigma(h)$. Use $\sigma(E)$ to describe the result of replacing any top-level Herbrand term h in E for which $\sigma(h)$ is defined by $\sigma(h)$. We say σ* is a safe renaming from p to q *just in case q is type-compatible with p, q is independent of p, and if f_A is a Herbrand function which depends on the associated formula A and $\sigma(f_A(X))$ is defined, then p is implicated in A and $\sigma(f_A(X)) = f_{\sigma(A[q/p])}(\sigma(X[q/p]))$.*

Observe that if σ safely renames from p to q, then since q is independent of p, $\sigma(q) = q$. Moreover since q is type compatible with p, $\Xi_X \triangleright J$ implies $\Xi_{\sigma(\theta(X))} \triangleright \sigma(\theta(J))$.

Lemma 24 (possibility of renaming) *We are given a closed relaxed Herbrand tableau, T, whose root carries the line L, such that no Herbrand term that occurs in L is introduced by a Herbrand inference in T and no general inference with instance x and no Herbrand rule introducing x lies on a path from the root to a Herbrand rule introducing x. Suppose T has a unary inference R at the root which applies to principal expression E to yield a side expression E':*

$$\cfrac{\Sigma_0 \triangleright \Gamma_0, E}{\cfrac{\Sigma_0' \triangleright \Gamma_0, E, E'}{T'}}\, R$$

Suppose some Herbrand term or Herbrand prefix p occurs in E' but is not implicated in the line $\Sigma_0 \triangleright \Gamma_0, E$. Now let q be type-compatible with p, independent of p, and such that the figure:

$$\frac{\Sigma_0 \triangleright \Gamma_0, E}{\Sigma_0'[q/p], \Delta \triangleright \Gamma_0, E, E'[q/p]}\, R'$$

instantiates a tableau inference figure for principal expression E. Then we can construct from T a new closed relaxed Herbrand tableau T_*, with the same root as T, containing corresponding inferences in a corresponding order to T, but in which p does not occur. Any unchecked symbol occurrences in T_* correspond to unchecked symbol occurrences in T, but if $\Xi_q \subseteq \Sigma'[q/p], \Delta$ then there are no unchecked occurrences of q in T_*.

The **proof** is by induction on the structure of closed relaxed Herbrand tableaux. For the induction hypothesis, we assume that if we have a subtableau of T of height

h whose root carries $\Sigma \triangleright \Gamma$ with $\Sigma_0'[q/p], \Delta \subseteq \Sigma$ and we have a safe renaming σ from p to q, then we can construct a corresponding subtableau whose root carries $\sigma(\Sigma[q/p]) \triangleright \sigma(\Gamma[q/p])$. The base case, for a closure inference, is straightforward: complementary formulas remain complementary under substitutions, so the inference remains a correct closure; since $\Sigma_0'[q/p], \Delta \subseteq \Sigma$, if $\Xi_q \subseteq \Sigma'[q/p], \Delta$, there are no unchecked occurrences of q in the closure.

So we assume a subtableau of height $h+1$ and consider case analysis on the inference at the root. Translation of boolean inference is immediate. For a general inference, such as necessity—

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}\Box_i A_X^{\mu}}{\Sigma \triangleright \Gamma, \mathbf{t}\Box_i A_X^{\mu}, \mathbf{t}A_{X,N,\nu}^{\nu}}$$
$$T'$$

—we obtain the revised proof T_*' by hypothesis and construct

$$\frac{\sigma(\Sigma[q/p]) \triangleright \sigma(\Gamma[q/p]), \sigma((\mathbf{t}\Box_i A_X^{\mu})[q/p])}{\sigma(\Sigma[q/p]) \triangleright \sigma(\Gamma[q/p]), \sigma((\mathbf{t}\Box_i A_X^{\mu})[q/p]), \sigma((\mathbf{t}A_{X,N,\nu}^{\nu})[q/p])}$$
$$T_*'$$

We need only that there is a derivation of $S, \Xi_{\sigma((X,N,\nu)[q/p])} \triangleright \sigma(\mu[q/p])/\sigma(\nu[q/p]) : i$; this follows from the side condition on the untransformed general inference and the fact that σ is a safe renaming.

Finally, consider a Herbrand inference such as possibility:

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}\Diamond_i A_X^{\mu}}{\Sigma, \mu/\mu\eta : i \triangleright \Gamma, \mathbf{t}\Diamond_i A_X^{\mu}, \mathbf{t}A_{X,\mu\eta}^{\mu\eta}}$$
$$T'$$

There are two cases. If p is not implicated in A, then $\sigma(A[q/p]) = A$ and we can construct

$$\frac{\sigma(\Sigma[q/p]) \triangleright \sigma(\Gamma[q/p]), \sigma((\mathbf{t}\Diamond_i A_X^{\mu})[q/p])}{\sigma((\Sigma, \mu/\mu\eta : i)[q/p]) \triangleright \sigma(\Gamma[q/p]), \sigma((\mathbf{t}\Diamond_i A_X^{\mu})[q/p]), \sigma((\mathbf{t}A_{X,\mu\eta}^{\mu\eta})[q/p])}$$
$$T_*'$$

(We use T_*' obtained by the induction hypothesis.) Otherwise, p is implicated in A; hence η takes the form $\eta_A(\ldots)$; we consider σ' exactly like σ except $\sigma'(\eta_A(\mu, X)) = \eta_{\sigma(A)}(\sigma'(\mu[q/p]), \sigma'(X[q/p])))$. By assumption on the form of the overall derivation T, $\eta_A(\mu, X)$ does not occur in the root; hence σ' remains a safe renaming. We can apply the induction hypothesis and construct:

$$\dfrac{\sigma(\Sigma[q/p]) \triangleright \sigma(\Gamma[q/p]), \sigma((\mathbf{t}\Diamond_i A_X^{\mu})[q/p])}{\sigma'((\Sigma, \mu/\mu\eta : i)[q/p]) \triangleright \sigma'(\Gamma[q/p]), \sigma'((\mathbf{t}\Diamond_i A_X^{\mu})[q/p]), \sigma'((\mathbf{t}A_{X,\mu\eta}^{\mu\eta})[q/p])} \atop T'_*$$

■

5.2 Specialized Closure Conditions for Modalities

We can now reformulate closure conditions for restricted modal logics, using Lemma 24 as a tool for rewriting symbols in relaxed Herbrand proofs.

First we observe, as hinted in section 5.1, that unchecked occurrences of first-order terms are not a problem. Define a condition $\Phi_F(\Sigma, \mathbf{u}A_X^{\mu})$ true just in case $\Xi_X \setminus \Sigma$ is a set of statements of the form $t : \mu$. Then in a relaxed Herbrand tableau that satisfies Φ_F, all unchecked occurrences of terms are occurrences of first-order terms.

Lemma 25 *Let T be a relaxed Herbrand tableau that satisfies Φ_F, with a root that carries no Herbrand terms and with the ordering property that no general inference with instance x and no Herbrand rule introducing x lies on a path from the root to a Herbrand rule introducing x. Then we can transform T into a corresponding Herbrand tableau T_* by substitutions of symbols.*

Proof. By induction on the number of unchecked occurrences of first-order symbols in T. If there are none, we in fact have a Herbrand tableau. Suppose the claim is true for T with n occurrences or fewer, and consider T with $n+1$. Consider any such symbol h, and consider an inference which introduces h on a branch containing unchecked occurrences of h, such that there are no other inferences which introduce h closer to the root. This must be a general inference; call it R and schematize it as in Lemma 24:

$$\dfrac{\Sigma_0 \triangleright \Gamma_0, E}{\Sigma'_0 \triangleright \Gamma_0, E, E'}\, R \atop T'$$

The conditions on T and R ensure that h is not implicated in the line $\Sigma_0 \triangleright \Gamma_0, E$—$h$ could be implicated here only if some rule closer to the root introduced h or if h was implicated in the root itself. So consider any constant symbol from the language c. c is type-compatible with h (since c is defined at all worlds); c is independent of h (since c contains no Herbrand symbols); and the figure below is a correct tableau inference:

$$\dfrac{\Sigma_0 \triangleright \Gamma_0, E}{[c/p]\Sigma'_0, \Delta \triangleright \Gamma_0, E, E'[c/p]}\, R'$$

The conditions of Lemma 24 apply to give a new proof with strictly fewer unchecked occurrences of first-order symbols. ■

Second, we observe that unchecked occurrences of modal terms are not a problem—provided that all modalities are serial (or reflexive). Define a condition $\Phi_S(\Sigma, \mathbf{u}A_X^{\mu})$ true just in case $\Xi_X \backslash \Sigma$ is a set of statements of the form $\mu/\mu\eta : i$.

Lemma 26 *Consider a regime where no modalities are assigned non-serial types K, KB, K4, K5 or K45. Let $\mathcal{T}$ be a relaxed Herbrand tableau that satisfies Φ_S with a root that carries no Herbrand terms and with the ordering property that no general inference with instance x and no Herbrand rule introducing x lies on a path from the root to a Herbrand rule introducing x. Then we can transform $\mathcal{T}$ into a corresponding Herbrand tableau $\mathcal{T}_*$ by substitutions of symbols.*

Proof. By induction on the number of unchecked occurrences of modal Herbrand symbols in $\mathcal{T}$. If there are none, we in fact have a Herbrand tableau. Suppose the claim is true for $\mathcal{T}$ with n occurrences or fewer, and consider $\mathcal{T}$ with $n+1$. There must be some symbol η on a branch that contains unchecked occurrences of η with the property that no other such symbol occurs closer to the root of the tableau than η. Consider the closest inference to the root on this branch which introduces η. This must be a general inference; call it R and schematize it as in Lemma 24:

$$\cfrac{\Sigma_0 \rhd \Gamma_0, E}{\begin{array}{c}\Sigma_0' \rhd \Gamma_0, E, E' \\ \mathcal{T}'\end{array}}\, R$$

The conditions on $\mathcal{T}$ and R ensure that η is not implicated in the line $\Sigma_0 \rhd \Gamma_0, E$—$\eta$ could be implicated here only if some rule closer to the root introduced η or if η was implicated in the root itself. The typing missing for η is some statement $\mu/\mu\eta : i$; the conditions on $\mathcal{T}$ and R also ensure that $\Xi_\mu \subseteq \Sigma_0$ as per the proof of Theorem 3. We consider two cases, depending on whether i is serial or reflexive. If i is serial, we can introduce an appropriate η' here so that the figure below instantiates the special necessity inference:

$$\frac{\Sigma_0 \rhd \Gamma_0, E}{\Sigma_0'[\eta'/\eta], \Delta \rhd \Gamma_0, E, E'[\eta'/\eta]}\, R'$$

η' is type-compatible with η; η' is independent of η since it is constructed from a different Herbrand function from and arguments that are independent of η.

Otherwise, if i is reflexive, we can simply replace the prefix $\mu\eta$ by the prefix μ, and retain a general inference. $\mu\eta$ is type-compatible with μ, since $\mu/\mu : i$ is derivable; μ is independent of η as guaranteed by the structure of $\mathcal{T}$ and the choice of R. For the same reason, $\Xi_\mu \subseteq \Sigma_0$.

In either case, the conditions of Lemma 24 now apply to give a new proof with strictly fewer unchecked occurrences of modal Herbrand terms. ∎

5.3 Streamlining Tracking Terms

The final optimizations depend on an observation about typing for modalities that are *neither* euclidean nor symmetric.

Lemma 27 (Prefix typing) *Consider a regime in which no euclidean or symmetric modalities (of classes KB, K5, K45, KD5, KD45 B or S5) occur. Then if* $\Xi_X \triangleright \mu/\nu : i$ *then* $\Xi_\nu \triangleright \mu/\nu : i$ *and* μ *is a prefix of* ν.

Proof. Observe that if μ is a prefix of ν then μ occurs in Ξ_ν and $\Xi_\mu \subseteq \Xi_\nu$. The base cases are the inference figures (K) and (T): the claim follows directly from the form of the judgments involved $\mu/\mu\eta : i$ (where μ must be a subterm of η) and $\mu/\mu : i$. The inductive cases are the (Inc) and (4) inference figures. (Inc) follows immediately from the induction hypopethesis. For (4), we have $\Xi_X \triangleright \mu/\mu' : i$ and $\Xi_X \triangleright \mu'/\nu : i$. We apply the induction hypothesis to the second; we obtain $\Xi_\nu \triangleright \mu'/\nu : i$ plus the fact that μ' occurs in Ξ_ν. This means $\Xi_{\mu'} \subseteq \Xi_\nu$ so by applying the induction hypothesis to the first component typing derivation, we get $\Xi_\nu \triangleright \mu/\mu' : i$ and μ occurs in Ξ_ν. Since μ is a prefix of μ' and μ' is a prefix of ν, μ is a prefix of ν. ∎

In regimes where no modalities are euclidean or symmetric, we now observe that we can use restricted general inference figures which do not permit free choice of tracking terms. For example, in place of the necessity figure:

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}\Box_i P^\mu_X}{\Sigma \triangleright \Gamma, \mathbf{t}\Box_i P^\mu_X, \mathbf{t}P^\nu_{X,N,\nu}}$$

we can now use the restricted necessity figure:

$$\frac{\Sigma \triangleright \Gamma, \mathbf{t}\Box_i P^\mu_X}{\Sigma \triangleright \Gamma, \mathbf{t}\Box_i P^\mu_X, \mathbf{t}P^\nu_{X,\nu}}$$

That is, the only change is the elimination of the choice of tracking terms N systematically in the necessity, extra special necessity and universal inferences.

Since the new inferences are restrictions of the usual Herbrand inferences, the Herbrand soundness theorem continues to apply; we need only adapt the Herbrand completeness theorem, Theorem 4, to show that ground proofs can be transformed into Herbrand proofs using these inferences.

Recall that the inductive hypothesis of the proof of Theorem 4 is that the typing Σ on each tableau line is rewritten to a Herbrand typing $\sigma(\Sigma)$ *for* the line. In rewriting general inferences, we have from the ground proof that any typing conditions are met—for example in the case of a necessity inference $S, \Sigma \triangleright \mu/\nu : i$. Therefore $S, \sigma(\Sigma) \triangleright \sigma(\mu)/\sigma(\nu) : i$. By Lemma 27, it follows that $S, \Xi_{\sigma(\nu)} \triangleright \sigma(\mu)/\sigma(\nu) : i$ and hence the restricted tracking of side formulas in the Herbrand proof suffices. ∎

5.4 Specialized Closure Conditions for Fragments

We can also exploit the prefix result to obtain an analogue of Lemma 2 of [Stone, 1999b], when no modalities are euclidean or symmetric and when no logical statements contain negation or possibility.

Lemma 28 (Irrelevance) *Let T be a relaxed Herbrand tableau with inferences ordered so that no Herbrand term that occurs in L is introduced by a Herbrand inference in T and no general inference with instance x and no Herbrand rule introducing x lies on a path from the root to a Herbrand rule introducing x. Suppose the root of T carries a line $\Sigma \triangleright \Gamma, \Gamma^*, \Delta$ where all the Γ and Γ^* expressions are of the form $\mathbf{t}E$ and the Δ expressions are of the form $\mathbf{f}E$, where no operators of possibility or negation occur in Γ, Γ^* or Δ and where for every $\mathbf{u}A_X^\mu \in \Gamma^*$ there is no $\mathbf{u}B_Y^\mu \in \Delta$ such that μ is a prefix of ν. Then T can be transformed into a relaxed Herbrand tableau of $\Sigma \triangleright \Gamma, \Delta$.*

The **proof** is a straightforward induction on the structure of proofs. A few key observations, based on Lemma 27 and the logical fragment, allow the proof presented in [Stone, 1999b] to go through. First, modal Herbrand inferences are associated only with negative statements, while modal general inferences are associated only with positive statements. Second, consider those Herbrand inferences, e.g.:

$$\frac{\Sigma \triangleright \Gamma, \Delta, \mathbf{f}\Box_i A_X^\mu}{\Sigma, \mu/\mu\eta : i \triangleright \Gamma, \Delta, \mathbf{f}\Box_i A_X^\mu, \mathbf{f}A_X^{\mu\eta}}$$

The order of inferences ensures that η is new, so if a formula is not associated with a prefix of μ, it's not associated with a prefix of $\mu\eta$ either. Finally, consider those general inferences, e.g.:

$$\frac{\Sigma \triangleright \Gamma, \Delta, \mathbf{t}\Box_i A_X^\mu}{\Sigma, \mu/\mu\eta : i \triangleright \Gamma, \Delta, \mathbf{t}\Box_i A_X^\mu, \mathbf{t}A_X^{\nu}}$$

The typing requires that μ be a prefix of ν so if μ is not a prefix of some Δ formula, then ν is not either. These observations allow inferences to Γ^* formulas to be inductively discarded. ■

Lemma 28 provides an alternative method to eliminate the modal side conditions on closure rules. We assume the streamlined tracking system of section 5.3, in which every tracked formula is the result of an explicit instantiation.

Lemma 29 *Let T be a relaxed Herbrand tableau, such that the root of T carries a line $\Sigma \triangleright \Gamma, \Gamma^*, \Delta$ where all the Γ and Γ are of the form $\mathbf{t}E$ and the Δ are of the form $\mathbf{f}E$, where no operators of possibility or negation occur in Γ, Γ^* or Δ and where for every $\mathbf{u}A_X^\mu \in \Gamma^*$ there is no $\mathbf{u}B_Y^\mu \in \Delta$ such that μ is a prefix of ν. Then T can be transformed into a ground proof of $\Sigma \triangleright \Gamma, \Delta$.*

Proof. Transform T to a reordered T^*, then apply Lemma 25 to eliminate unchecked first-order terms. We next apply Lemma 28 to eliminate certain formulas and inferences from the proof. I claim that the resulting relaxed Herbrand tableau has no unchecked modal terms either. By the form of tracking, any untracked modal Herbrand term occurrence η is associated with some general inference R at which η is the instantiated parameter. Because the occurrence is untracked, there can be no corresponding Herbrand inference involving η closer to the root. This in turn means the inference R must involve a positive side expression $\mathbf{t}A_X^{\mu\eta}$ where $\mu\eta$ is not a prefix of ν for any negative expression $\mathbf{f}B_Y^{\nu}$. This is a impossible, for such an inference R would have been eliminated from the tableau by the application of Lemma 28. ■

5.5 A final observation

Consider either of the systems studied in Section 5.2 or 5.4, in which the side condition on closure rules are eliminated. We conclude by observing that such systems allow the form of sequents themselves to be streamlined. Construct a *typeless* version of these calculi in which tableau lines take the simple form Γ, where Γ is (as always) a multiset of signed tracked prefixed formulas—here we eliminate the typing Σ from the sequent. The tableau rules for the typeless calculi are obtained likewise by simply omitting Σ (and changes to Σ) from the formulation of the original inference rules.

The typeless calculi are sensible because the side condition on tableau rules—on the closure inference particularly—no longer depend on Σ in the original calcluli. The side conditions on inference rules therefore remain unchanged by the simplification to the typeless system.

It is therefore straightforward to show by induction that the typeless calculi are equivalent in provability to the original systems. To transform an original tableau to a typeless tableau, one simply erases the typing Σ on each tableau line in the tableau. Conversely, starting from a typing Σ and a typeless tableau, one simply inductively redecorates the tableau with typings derived from Σ according to the original tableau rules.

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