Fundamental Properties of Harmonic Bounding *

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1. Introduction

Evaluation in OT adjudicates competitions between linguistic structures, but it calculates only with the array of violations that the structures incur under each constraint: their *violation profiles*. Structures with the same profile are indistinguishable, and differences in structure only register to the extent that they correlate with differences in violation. General properties that govern relations between profiles in the space of all possible profiles will thus be inherited by any specific candidate set, even though actual structures may be distributed sparsely or asymmetrically in violation space.

Shifting the focus of inquiry from candidates in the space of linguistic forms to violation profiles in violation spaces will provide tools useful in analysis, computation, and learning. Of particular significance are the principles determining which profiles can never be optimal under any ranking, given that certain other profiles are known to be realized in the candidate set. (These never-optimal profiles we will call *losers*; by *winners* we mean the complement set of profiles optimal under some ranking.) The value of knowing the loser-*vs*.-winner status of a structure is manifest in many applications, especially (at the risk of paradox!) prior to conducting a specific competition. Consider the procedures involved in constructing candidates to test for optimality: given the mere existence of a candidate with a certain profile, knowledge of what it excludes under any ranking will render unnecessary the labor of constructing and evaluating candidates that are always defeated by it. Identifying loser profiles will eliminate improper learning targets and help determine what abstract structure ought be assigned to the observables; for example, we can avoid formally possible but perpetually suboptimal foot-parses for observed sequences of stressed and unstressed syllables, pruning subversive hypotheses (cf. Tesar 2000). Excluding losers is also essential to the analyst, who must know whether the set of competitors under consideration — inevitably finite — mistakenly omits some potentially optimal structures. A precise characterization of the regions of profile space defeated by the identified competitors will address this danger.

In Samek-Lodovici & Prince (1999) we show that every loser is harmonically bounded by some non-empty set of candidates. Here we shift perspective and investigate how to characterize the set L(A) of profiles turned into losers by a given profile set A. *A priori*, this set contains any profile bounded by any non-empty subset of A, included the potentially infinite set of losers collectively bounded by profiles ganging together so as to each beat the loser on some of the possible constraint rankings while leaving none uncovered. We will show that any set of losers collectively bounded by a profile set A is equivalent to the set of losers bounded by a single designated minimal profile directly identifiable from our knowledge of A, and which will call the 'bounding minimum' of A. Consequently, L(A) itself becomes determinable on the basis of *simple harmonic bounding* via bounding minima alone, simplifying aspects of its computation. The result is fully general, and applies to any set of profiles A, independently of whether the members of A are all winners, or some of them are themselves turned into losers by some other fellow profiles.

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We will start by laying out the basic properties of profile spaces and explaining how collective bounding reduces to simple bounding through such bounding minima. We then generalize this basic observation to any set of profiles A, showing how the corresponding set of losers L(A) can be systematically characterized in terms of simple bounding. We conclude with observations on computational complexity issues, and provide in the Appendix the formal proof of a couple of results presented informally in the text.

2. Violation Spaces

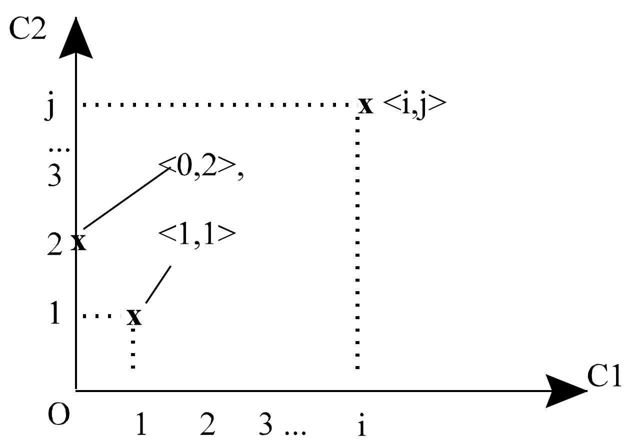
We are used to thinking of optimization as selecting the structures that best satisfy a ranked set of constraints among an infinite set of competing structures. As noted, however, optimization examines only the violation profiles of the competing structures, and not the structures themselves. For example, the 'perfect' profile, with zero violations on all constraints, if realized, is always optimal, independent of the structure it may correspond to in any particular analysis.

The proper domain for studying optimization is thus the space of all possible violation profiles rather than that of the corresponding structures. We will therefore represent a violation profile, or 'profile' for short, as a vector α, where the i^{th} coordinate value '$\alpha(i)$' represents the number of violations for α on the i^{th} constraint of some constraint set Σ. For example, if Σ consists of two constraints C_1 and C_2, the profile $\langle 0,2 \rangle$ satisfies C_1 and violates C_2 twice; the profile $\langle 1,1 \rangle$ violates C_1 and C_2 once each; the profile $\langle 100,0 \rangle$ violates C_1 one hundred times while satisfying C_2.

(1) **Def. Profile.** For any $\Sigma=\langle C_1,C_2,..,C_n \rangle$, a profile α over Σ is defined as the vector $\alpha=\langle \alpha(1) ,..., \alpha(n) \rangle$, where $\alpha(i)$ records the number of violations for constraint C_i.

We may conceive profiles as points in the *n*-dimensional violation space V^n —or V for short— determined by the Cartesian product of the *n* constraints in Σ. Each constraint constitutes an axis of V, allowing from zero to any number of violations for that constraint; we will use the terms 'constraint', 'dimension', and 'coordinate' interchangeably. The vector representing a profile spells out the coordinates of the point representing the profile in the violation space. For example, the figure below shows the position for the two profiles $\langle 0,2 \rangle$ and $\langle 1,1 \rangle$ examined above, as well as that for a generic profile $\langle i,j \rangle$ violating C_1 exactly *i* times and C_2 exactly *j* times.

(2) Violation space V^2 for $\Sigma=\{C_1, C_2\}$.



Optimization follows the lexicographic order that matches a chosen specific constraint ranking. Profiles with lower coordinate values on high-ranked constraints thus beat any other profile with higher values on those same constraints, including those performing better on lower-ranked ones. For example, when C_1 outranks C_2, profile $\langle 0,2\rangle$ precedes in the lexicographic order —and hence 'beats'— profile $\langle 1,1\rangle$, because it outperforms it on the highest constraint C_1. Profile $\langle 0,1000\rangle$ would beat $\langle 1,1\rangle$ too, again due to its lower C_1-coordinate, and for the same reason that *'az'* precedes *'ba'* in the lexicographic order used in dictionaries. Unlike dictionaries, however, OT allows ranking permutation, and on the ranking C_2>>C_1, the profile $\langle 1,1\rangle$ precedes both $\langle 0,2\rangle$ and $\langle 0,1000\rangle$, as well as any profile $\langle 0, j\rangle$ with $j>1$.

Before turning to our main goal, it is worth examining the relation between profiles and actual competing structures generated by GEN in linguistic analyses. Every generated structure necessarily corresponds to a profile in V, namely the one recording its violations on each constraint. Distinct structures may also share the same profile when they violate the same constraints the same number of times (see for example Grimshaw's analysis of optional complementizers, 1997). Structures related in this way will share the same optimization fate under all rankings, since this is determined by the profile alone. Some profiles, on the other hand, may lack a corresponding structure generable by GEN. For example, in most analyses, the perfect profile $O=\langle 0,0,...,0\rangle$, that satisfies all constraints and hence forms the origin of the violation space V, has no structural correspondent because constraints conflict with each other, and the satisfaction of one constraint entails the violation of another.[1]

It follows that the set of competing structures from actual OT analyses almost always corresponds to a subset of the possible profiles represented within a violation space. This provides a strong reason for studying optimization and harmonic bounding within violation spaces rather than the candidate sets provided by GEN, because any fundamental property of profile optimization in V also holds of optimization within specific candidate sets, whereas the converse need not be true.

In the following, we will use properties and theorems from Samek-Lodovici and Prince (1999). Although they do not mention violation spaces, they were established with respect to fully

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[1] The scenario where every profile corresponds to a competing structure, requires constraints with an infinite number of ordered strata, where each stratum is itself non-finite. For example, in V^2 this requires the two constraints shown below, with strata ordered by descending order as violations increase. Each stratum is infinite, yet each candidate has a distinct violation profile, shown in the subscripts. For example, structure c_{01} occurs in the first stratum of C_1 and the second of C_2, and corresponds to the profile $\langle 01\rangle$. Despite its complexity, the whole space has a unique winner which bounds all other structures: the perfect profile c_{00} which satisfies both constraints.

(1) Exhaustive scenario for V^2.

$$\begin{array}{cc} C_1 & C_2 \\ c_{00}, c_{01}, c_{02}, \ldots & c_{00}, c_{10}, c_{20}, \ldots \\ | & | \\ c_{10}, c_{11}, c_{12}, \ldots & c_{01}, c_{11}, c_{21}, \ldots \\ | & | \\ c_{20}, c_{21}, c_{22}, \ldots & c_{02}, c_{12}, c_{22}, \ldots \\ \ldots & \ldots \end{array}$$

abstract candidates allowing for any conceivable violation profile, and therefore concerning any conceivable point in V. Unlike arguments concerning specific candidate sets, they thus straightforwardly carry over to violation spaces, preserving their validity in full.

3. Harmonic Bounding in Violation Spaces

In this section, we define harmonic bounding, simplifying the definition of Samek-Lodovici and Prince (1999). We then show that the set of losers *collectively* bounded by a set of profiles can be characterized in terms of *simple* bounding alone. This is a useful result, because unlike collective bounding, simple bounding is particularly easy to test, as it only requires checking that the bounded profile be not better than its bounder on any coordinate. With this, we take the first step toward a complete characterization through simple harmonic bounding alone of the infinite loser set L(A) determined by any set of profiles A.

3.1 Defeating Sets

Central to the characterization of bounding is identifying a set of profiles that collectively prevents a bounded profile from being optimal under any ranking. Several ways of delimiting such sets are available; here we put forth one that proves particularly useful. A profile set will be said to constitute a *defeating set* for a profile λ if and only if it satisfies the property of 'reciprocity', which holds whenever it is not possible for λ to have less violations than a member of A on a constraint without some other member of A coming to the rescue by posting less violations than λ on that same constraint. The name 'reciprocity' reminds us that the members of A may reciprocate the rescuing action, with profiles acting as rescuers on one constraint while endangered by λ on another. The definition applies even when reciprocity is satisfied vacuously, and therefore a loser simply bounded by each member of A counts as defeated by A. Note that reciprocity is also vacuously satisfied for a singleton set A={α} relative to its unique member. The condition that λ be excluded from A ensures in this case that a defeater α is never grouped with the profiles it defeats.

(3) **Def. Defeating Set.** Let A be a non-empty profile set, and λ a profile in V but not in A. Then A is a defeating set for λ if and only if A satisfies the reciprocity condition below:

$$\textit{Reciprocity}: \forall i,\ i \leq n,\ \forall \alpha \in A,\ [\ \lambda(i) < \alpha(i) \Rightarrow \exists \alpha' \in A,\ \alpha'(i) < \lambda(i)\ ].$$

We also define the set D(A) of profiles defeated by a set A. These are all and only the profiles for which A acts as defeating set. We use the term 'defeated' rather than 'bounded' because as we will see shortly a profile might be harmonically bounded by a proper subset of A, and therefore be defeated by A, and yet not be bounded by A itself. We will also write '$A \sqsubset \lambda$' to indicate that λ is defeated by A, and '$\alpha \sqsubset \lambda$' to indicate that the singleton set A={α} defeats λ, in which case, as we will see, λ is simply bounded by α.

(4) **Def. D(A).** Let A be a profile set in V, then D(A) contains any profile λ in V that has A as defeating set.

$$D(A) = \{\lambda : \lambda \in V \text{ and A a defeating set for } \lambda\}.$$

Defeating sets constitute a superset of bounding sets, which according to the definition in (Samek-Lodovici & Prince 1999:46) must satisfy *reciprocity* and *strictness*, with strictness requiring bounders to beat a bounded profile λ on at least one constraint. The definition of bounding set adapted to violation spaces is provided below.

(5) **Def. Bounding Set *(Violation Spaces)*.** A profile set B in V is a bounding set for a profile *z* in V if and only if B has the following properties:

Strictness: $\forall\beta\in B, \exists i, i\leq n, \beta(i)<z(i)$.

Reciprocity: $\forall i, i\leq n, \forall\beta\in B, [\ z(i)<\beta(i) \Rightarrow \exists\beta'\in B, \beta'(i)<z(i)\ ]$.

For example, the set $A=\{\langle 2,3\rangle\}$ is a bounding set for $\lambda=\langle 3,3\rangle$, with its only member bounding λ via strictness on constraint C_1 and reciprocity vacuously. It is also a defeating set for λ, because reciprocity is satisfied. However, the set $A'=\{\langle 2,3\rangle, \langle 4,3\rangle\}$ is not a bounding set for λ because its second member does not satisfy strictness, even though A' satisfies reciprocity and therefore constitutes a defeating set for λ.

The most important property of bounding sets follows from the Bounding Theorem (Samek-Lodovici & Prince, 1999:11,47), which tells us that any profile with a non-empty bounding set is necessarily a loser, and any loser is necessarily bounded by some non-empty bounding set. A version of the theorem adapted to violation spaces is given below. The term 'W(K, Σ)' represents winners, i.e. all the profiles within a profile set K that are optimal under some ranking of the constraints in Σ, and the term 'B(x)' stands for 'bounding set for x'.

(6) **Bounding Theorem *(Violation Spaces)*.** Let Σ be a set of constraint coordinates for V. For any profile set K and profile λ in V, λ is suboptimal in K under any ranking of the constraint-coordinates Σ iff there is in K a non-empty bounding set $B(\lambda)$ for λ.

$$\lambda\notin W(K, \Sigma) \Leftrightarrow B(\lambda)\neq\emptyset.$$

Crucially, in spite of the weaker condition defining them, defeating sets inherit from bounding sets the crucial property of turning any profile that they defeat into a loser whenever they are not empty.

That any loser is defeated by some non-empty defeating set follows by the bounding theorem, because the loser must be bounded by some non-empty bounding set, and every non-empty bounding set is a defeating set because it satisfies reciprocity. Conversely, a non-empty defeating set A for some profile λ always includes a non-empty bounding set B for λ, thus turning it into a loser due to the bounding theorem. In this case, the relevant bounding set B for λ can be built by collecting all members of A that post less violations than λ on some constraint, thus satisfying strictness. Note that B is necessarily non-empty, since by definition λ does not belong to A, and therefore must differ from each member of A on some constraint C. If λ beats one or more of them on some constraint C, reciprocity on A ensures that some other member β beats λ on C, and is collected in B. On some constraint C$'$, the subset B might satisfy reciprocity vacuously, with λ neither beating nor beaten by them. Overall, B satisfies the definition for bounding set and thus ensures the loser status of λ through the bounding theorem, which is therefore inherited by A.

The properties of defeating sets just illustrated are formally stated in the lemma and theorem below, whose formal proof we postpone to appendix A in order to more speedily proceed toward our main goal. The Defeating-Bounding Lemma expresses the fact that every non-empty defeating set includes a non-empty bounding set and vice versa, and the Defeating Theorem formally records the fact that a profile with a non-empty defeating set is necessarily a loser. (The symbol '$\sqsubset$' stands for the defeating relation introduced above.)

(7) **Lemma. Defeating-Bounding.** Let A be a profile set and λ a profile in V. Then A is a non-empty defeating set for λ if and only if there exists a non-empty set B in A that constitutes a bounding set for λ.

$$\forall A\neq\emptyset,\ A\sqsubset\lambda \Leftrightarrow \exists B\subseteq A,\ B\neq\emptyset,\ B=B(\lambda).$$

(8) **Defeating Theorem.** Let Σ be a set of constraint coordinates for V, and let K be a profile set in V and λ a profile in K. Then λ is suboptimal in K under any ranking of Σ iff there is in K a non-empty defeating set A for λ.

$$\lambda\notin W(K, \Sigma) \Leftrightarrow\ \exists A\neq\emptyset,\ A\subseteq K,\ A\sqsubset\lambda.$$

By the defeating theorem, any profile defeated by a non-empty set A is a loser, because it is bounded by some non-empty subset of A. The set D(A) thus collects the profiles that are losers because defeated by A.

3.2 Defeating via Simple Harmonic Bounding

We may now turn our attention to the set L(A) collecting *all* losers for a specific set of profiles A, i.e. including all those profiles that are beaten across all available rankings by one or more the members of A. We will show that L(A) coincides with the union of all profiles defeated by some non-empty subset of A. We will then study the relation between L(A), D(A), and simple harmonic bounding in the case where A contains only one profile, which will later enable us to reduce the computation of D(A), and therefore of L(A), to checking a collection of simple harmonic bounding relations.

Consider the set L(A), defined below, collecting *any* profile turned into a loser when competing against A, including those defeated by a subset of A but not by A itself.

(9) **Def. L(A).** Let A be a profile set and λ a profile in V. Then λ belongs to the set L(A) of losers for A if and only if λ does not belong to W(A∪{λ}, V).

$$L(A) = \{\lambda : \lambda\in V \text{ and } \lambda\notin W(A\cup\{\lambda\}, V)\}.$$

The profiles defeated by some proper subset of A but not by A itself are not in D(A). Consider for example the two profiles $\alpha=\langle 2,0\rangle$ and $\beta=\langle 0,2\rangle$. The profile $\lambda=\langle 0,4\rangle$ is defeated by B={β}, with reciprocity applying vacuously because λ is never better than β. Yet, λ is not defeated by $A=(\alpha, \beta)$ because reciprocity does not hold: on C_1, λ is better than α but not strictly worse than β as reciprocity requires. Thus λ is not in D(A) even if it is in L(A) because it is defeated by the subset B.

The appropriate relation between L(A) and D(A) thus requires L(A) to collect any profile defeated by some non-empty subset of A. This relation is formalized in the lemma below, and an example will follow in the following section.

(10) **Lemma. Loser Set.** For any set A in V, its loser set L(A) consists of the union of all λ defeated by some non-empty subset A' of A.

$$L(A) = \cup_{A' \subseteq A,\, A' \neq \emptyset} D(A').$$

Pf. - Let us prove $L(A) \subseteq \cup_{A' \subseteq A,\, A' \neq \emptyset} D(A')$.

Let $\lambda \in L(A)$. Since λ is a loser against A, by the bounding theorem A includes a non-empty bounding set A' for λ. Then by the defeating bounding lemma, B is a defeating set for λ and $\lambda \in D(A')$.

- Let us now prove $\cup_{A' \subseteq A,\, A' \neq \emptyset} D(A') \subseteq L(A)$.

By hypothesis $\exists A' \subseteq A$, $A' \neq \emptyset$, $\lambda \in D(A')$. Therefore, by the defeating-bounding lemma, there is a non-empty bounding set $B \subseteq A' \subseteq A$ bounding λ. It follows that there is a non-empty bounding set B for λ in A, and therefore $\lambda \notin W(A \cup \{\lambda\}, V)$. Therefore $\lambda \in L(A)$. □

The only case in which L(A) and D(A) necessarily coincide occurs when A is a singleton. In this case, the only available subsets of A are the empty set Ø and A itself. But losers require non-empty defeating sets, and hence only A qualifies. All members of L(A) will then be collected in D(A).

When a defeating set consists of just one element α, reciprocity may only be satisfied vacuously, as non-vacuous satisfaction always requires at least two members in a set. However, it prevents any defeated profile λ from incurring less violations than α on any coordinate, because this would violate reciprocity, since no other profile is available to rescue α. In order to be distinct from α, any defeated profile λ will also have to differ from α on at least one coordinate. As a result, α will present less violations than λ on at least one coordinate, and never have more violations than λ on any other. In other words, α simply harmonic bounds λ in the manner first discussed in Prince & Smolensky (1993), and the defeating relation coincides with that of simple harmonic bounding. The defeating singleton lemma below formalizes this result.

(11) **Lemma. Defeating Singleton**. Let α be a profile in V. Then the set $D(\alpha)$ of profiles defeated by the singleton set $A=\{\alpha\}$ coincides with the set of profiles simply bounded by A.

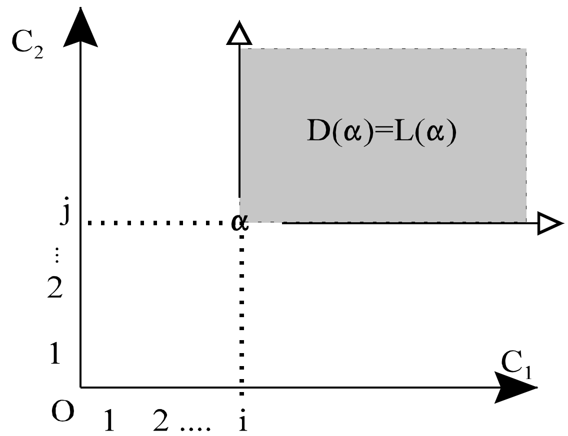
$$D(\alpha)=\{\lambda: \lambda \in V, \forall i\ \alpha(i) \leq \lambda(i) \text{ and } \exists j\ \alpha(j) < \lambda(j)\}.$$

Pf. Let B be the set of profiles simply bounded by α, i.e. $B=\{\lambda: \lambda \in V, \forall i\ \alpha(i) \leq \lambda(i), \exists j\ \alpha(j) < \lambda(j)\}$. $B \subseteq D(\alpha)$ is trivial, as any bounding set satisfies reciprocity and hence qualifies as defeating set as well.

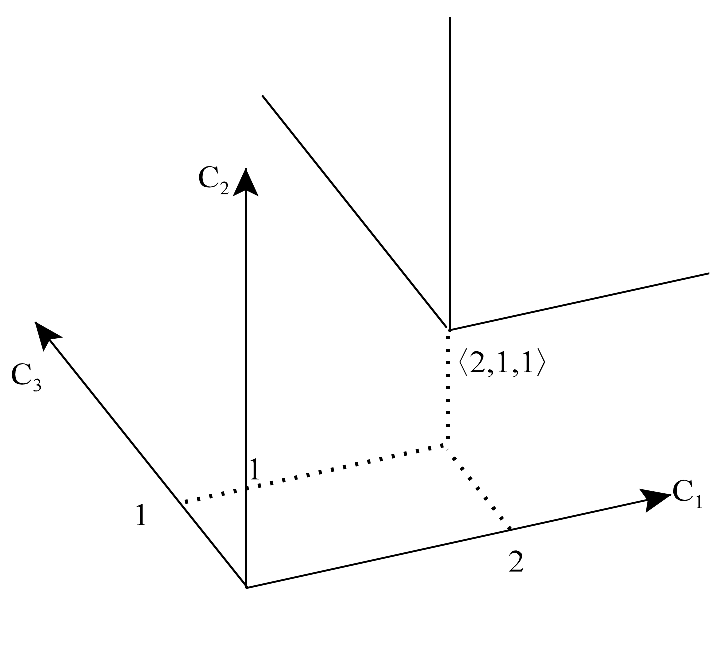
As for $D(\alpha)\subseteq B$, assume $D(\alpha)\neq\emptyset$, then by the defeating bounding lemma, for any $\lambda\in D(\alpha)$ there is a non-empty B qualifying as a bounding set for λ, and since A is a singleton, B=A. □

Simple bounding has a straightforward geometrical characterization in V^2. The set $D(\alpha)$ of profiles defeated — and hence simply bounded — by a singleton set $A=\{\alpha\}$ covers the infinite region delimited by two half-lines starting at α and parallel to the coordinate axes, as shown in the figure below. The shaded region, with the exclusion of α, contains all the profiles in $D(\alpha)$, and in this case it also characterizes the loser set $L(\alpha)$, since with singletons the two sets coincide. Figure (13) shows the region of defeated profiles in a three-dimensional system.

(12) $D(\alpha)$ in V^2 for $\alpha=\langle i,j\rangle$.



(13) $D(\alpha)$ in V^3 for $\alpha=\langle 2,1,1\rangle$.



Simple bounding shows a series of important properties that will later permit us to characterize defeating sets in terms of simple bounding. To begin with, simple bounding is transitive, as recorded in the lemma below. This property will later enable us to show that the defeating relation is itself transitive and idempotent.

(14) **Lemma. Bounding Transitivity**. Let α, β, and γ be profiles in V. Then, if γ is in $D(\beta)$ and β is in $D(\alpha)$, then γ is in $D(\alpha)$ as well.

$$\forall\alpha, \beta, \gamma\in V\ [\ \alpha\sqsubset\beta\ \&\ \beta\sqsubset\gamma\Rightarrow \alpha\sqsubset\gamma\ ]$$

Pf. By the lemma on defeating singletons, $\forall i\ \alpha(i)\leq\beta(i)$ and $\forall i\ \beta(i)\leq\gamma(i)$, therefore $\forall i\ \alpha(i)\leq\beta(i)\leq\gamma(i)$. Further, by defeating singleton $\exists j\ \beta(j)<\gamma(j)$, therefore $\exists j\ \alpha(i)\leq\beta(i)<\gamma(i)$. It follows that $\alpha\sqsubset\gamma$. □

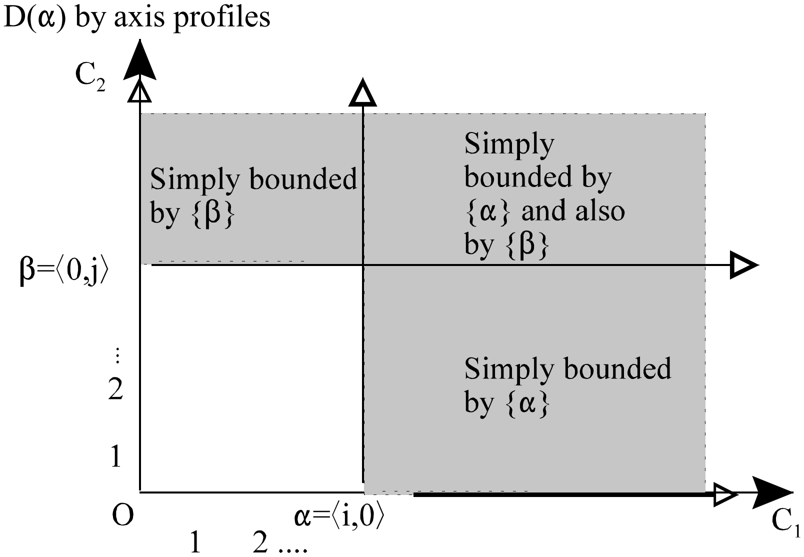
Another important property of simple bounding is that the sum of the coordinates of each defeated profile is always greater than that of their defeater. This follows from the property —forced by reciprocity in the way described above— that any defeated profile may at most equal the coordinates of the defeater on all axes but one, eventually yielding a greater coordinate sum. The converse does not hold; for example, the profile $\langle 0,100\rangle$ is not bounded by the profile $\langle 1,0\rangle$, even if the latter has a lower coordinate sum. The correct entailment is formalized below.

(15) **Lemma. Coordinate Sum**. For any α and β in V, if $\alpha\sqsubset\beta$, then the sum of the coordinates for β exceeds that for α:

$$\forall\alpha, \beta\in V,\ \alpha\sqsubset\beta \Rightarrow \Sigma_i\ \beta(i) > \Sigma_i\ \alpha(i).$$

A third interesting property is that when *n* profiles lie on the *n* distinct axes of V, the union of the sets $D(\alpha)$ projected by each profile covers the entire space V except for a finite part of it. For the two dimensional case, the result is illustrated by the figure below: the only profiles not defeated by either $\alpha=\langle i,0\rangle$ or $\beta=\langle 0,j\rangle$ are those in the white region of the plane V^2. In actual analyses, however, it is unlikely that each axis hosts the profile of candidates generated by GEN, because lying on an axis requires a zero coordinate on all other axes, i.e. satisfaction of all other involved constraints.

(16) Union of $D(\alpha)$ by axis profiles



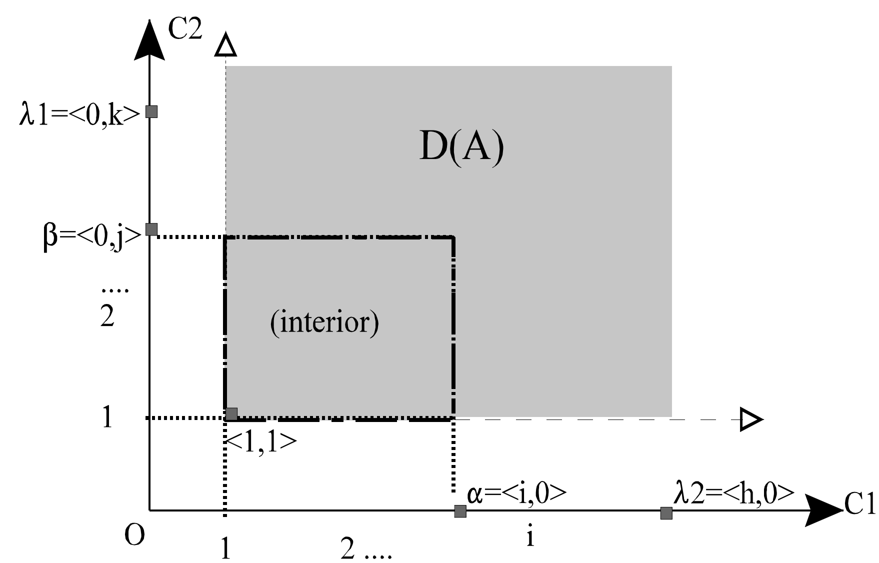
The profiles in the shaded regions belong to either D(α) or D(β) or both, and thus are also included in L(A), where A={α, β}. However, they do not exhaust L(A), because the profiles collectively bounded by α and β via non-vacuous reciprocity are still missing. These are examined in the next section.

3.3 Collective Defeating through Reciprocity

When a defeating set A contains two or more profiles, the set of defeated profiles D(A) need not match the union of the profiles defeated by each of its members.

First of all, losers in L(A) sharing a coordinate value with some defeater α but not others might be in D(α) while not being in D(A). Consider for example the two-dimensional case from the previous section with A={α, β} where α=⟨i,0⟩, and β=⟨0,j⟩}. As we saw there, a loser λ_1=⟨0,k⟩ is simply bounded by β=⟨0,j⟩ whenever *k* is greater than *j*. Yet, A={α, β} does not qualify as a defeating set for λ_1 even if it includes β. The reason is that λ_1 beats α=⟨i,0⟩ on C_1, and hence reciprocity kicks in, requiring A to host a profile that beats λ_1 on that same coordinate. But A includes none, because β shares the *same* value as λ_1 on C_1, namely zero. Any profile like λ_1 will thus be in D(β) but not in D(A), even though β is a member of A. The same of course holds true for the same reasons for any symmetric loser λ_2 sharing the minimal C_2 coordinate value of α. As we will show later, this is a fully general property: losers in L(A) sharing a coordinate value *v* with some members of a profile set A but not others are not in D(A) whenever *v* is minimal across the members of A.

(17) Profiles in D(A), with A={α, β}.



A second mismatch concerns the additional profiles defeated by A through non-vacuous reciprocity, which are not necessarily defeated by any individual member. In the figure above, the shaded region represents D(A) and includes all the profiles defeated by A={α,β}. The profiles in the lower 'interior' box are not simply bounded by either α or β, but only collectively by the two profiles together through reciprocity. This region includes profiles such as ⟨1,1⟩, which beats α on C_1, and β on C_2, and hence could not be simply bounded by either α or β alone. The interior thus constitutes the profile contribution made by D(A) to the loser set L(A) that is not already available through the profiles simply bounded by the distinct members of A.

Some interesting properties of interiors already emerge from the above figure. First of all, the coordinate values *i* and *j* of α and β could be arbitrarily large and they would still defeat any profile in the interior, including $\langle 1,1 \rangle$. This because optimization selects the profiles with the least violations according to the lexicographic order imposed by each constraint ranking. Even with *i* and *j* set to 1000, optimization on the ranking $C_1 >> C_2$ selects profile β over any profile in the interior because β is minimal on constraint C_1, where it has zero violations. Symmetrically, optimization over $C_2 >> C_1$ selects α because it posts zero violations on C_2.

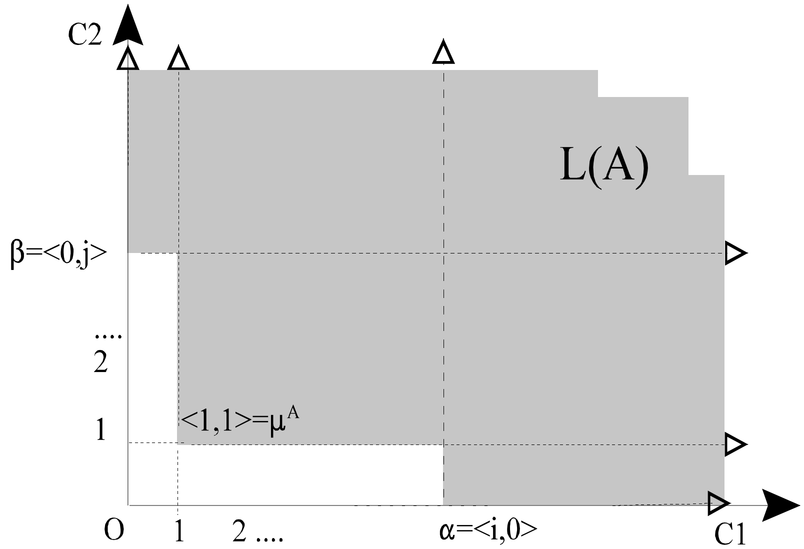
Furthermore, ganging together permits α and β to defeat profiles whose coordinate sum is lower than theirs, making inroads toward the origin $O=\langle 0,0 \rangle$. This occurs because for every coordinate C the only value that matters is the minimal one available within $A=\{\alpha, \beta\}$. Any profile with higher values will lose to the member of A with the minimal C-coordinate on any ranking placing C highest. The profiles in the interior can have low coordinates across the board, yielding an overall lower coordinate sum than their defeaters, but will crucially be always worse than some defeater on any coordinate, as dictated by Reciprocity.

Note that although the interior is finite in this specific example, it need not be in the general case. To see this, we have to move to V^3 or higher spaces. Consider for example the bounding set $A=\{\alpha, \beta\}$, where $\alpha=\langle 0,2,5 \rangle$, and $\beta=\langle 2,0,5 \rangle$. Any profile $\lambda=\langle 1,1,5+i \rangle$ with $i \geq 0$ beats α on C_2 and β on C_1, and therefore it is simply bounded by neither α nor β. Yet, it is defeated by A when α and β cooperate via reciprocity, with λ beaten by α on any ranking with C_1 highest, and by β on those with C_2 highest . Profile λ thus belongs to the interior, and this remains true whatever the value of *i*. Therefore the interior in this case contains infinitely many profiles.

The most important property concerns the entire set of defeated profiles D(A). This set always allows for a minimal element with the least coordinate sum that turns out to simply bound any other profile in D(A), and is henceforth called the 'bounding minimum'. In the two-dimensional case shown in figure (17) above, the bounding minimum is $\mu^A=\langle 1,1 \rangle$, and can be seen to simply bound the whole of D(A).

We may now exploit bounding minima to reduce the set of defeated profiles D(A) to that simply bounded by the bounding minimum of A, which in turn is collectively defeated by A. As we will see shortly, the coordinates of the bounding minimum follows straightforwardly from those of the members of A. This result will in turn ease the computation of the overall loser set L(A), which will match the union of all D(B) built from some subset B of A, with each D(B) easily computable in terms of simple bounding via its corresponding μ^B. This will free us from the need to ever compute the collective defeating relations based on reciprocity. For example, in the two-dimensional case examined so far, the set L(A) coincides with the union of the profile set simply bounded by μ^A, plus the sets simply bounded by the minima of the proper subsets of A, which in this case coincide with the sets of profiles simply bounded by α and β. The loser set L(A) thus coincides with the union of the three infinite regions originating at α, β, and μ^A shown below, matching the entire shaded region in the figure below with the exclusion of α and β.

(18) L(A) for A={α, β}



3.4 Bounding Minima

Let us now examine the properties illustrated above from a more formal point of view, providing the associated theorems and demonstrations.

We start considering an important partition induced on the set of coordinates by any profile set A. For every coordinate, the members of A may or may not share the same number of violations. Therefore A induces a partition into a subset M^A of 'minimal' coordinates containing those whose value is shared across all members of A and is therefore minimal in A, and the set R^A of 'reciprocity' coordinates formed by all other coordinates, i.e. those for which at least two members of A disagree in their value. Incidentally, note that if A is a singleton, all coordinates are in M^A, and R^A is necessarily empty. The definition of M^A and R^A is provided below.

(19) **Def. Minimal and Reciprocity Coordinates.** Let V be determined by a set of constraints Σ, and let A be a set of profiles in V. Then the corresponding sets M^A and R^A of minimal and reciprocity coordinates are respectively defined as (i) the set of coordinates in V where all members of A are order-equivalent, and (ii) its complement.

(i) Minimal coordinates: $M^A = \{i : i \in \Sigma, \forall \alpha,\beta \in A, \alpha(i)=\beta(i)\}$,
(ii) Reciprocity coordinates: $R^A = \Sigma - M^A = \{i : i \in \Sigma, i \notin M^A\}$.

We may now build the bounding minimum μ^A for a set A on the basis of M^A and R^A. The minimum is the profile with minimal coordinate sum collectively defeated by the members of A via reciprocity,[2] and simply bounds any other profile defeated by A. As we will see shortly, this latter property requires that μ^A shares any minimal coordinate value shared by all members of A on M-coordinates, and any other minimal value available across the members of A incremented by one for R-coordinates. For example, if A consists of $\alpha=\langle 0,7,5\rangle$ and $\beta=\langle 4,2,5\rangle$, then μ^A will share with the members of A the five violations on C_3, and post one more violation than the minimal value available in A for C_1 and C_2, yielding $\mu^A=\langle 1,3,5\rangle$. The coordinate values for the bounding minimum are recorded in the definition below.[3]

(20) **Def. Bounding Minimum**. Let A be a set of profiles in V, and M^A and R^A the corresponding coordinate partition. The corresponding bounding minimum μ^A is then defined as follows:

$$\forall i \in M^A\ \mu^A(i)=\alpha(i) \text{ for any } \alpha \in A,$$
$$\forall i \in R^A\ \mu^A(i)=\alpha_{min}(i)+1, \text{ where } \alpha_{min} \in A \text{ and } \forall \alpha' \in A\ \alpha_{min}(i) \leq \alpha'(i).$$

As mentioned, the virtue of μ^A is that any profile λ defeated by A other than μ^A itself is guaranteed to be simply bounded by μ^A. This is proven in the following theorem, which also clarifies why μ^A has the coordinates defined above. Intuitively, if a defeated profile λ beats some α in A on some R-coordinate *i*, then by reciprocity some other α' in A must beat λ on *i*. In particular, if α_{min} is the member of A with the minimal value for *i*, then $\lambda(i)$ must be higher than $\alpha_{min}(i)$, else reciprocity would be violated. The minimum μ^A, set to $\alpha_{min}(i)+1$, thus presents the minimal possible value on any R-coordinate *i* compatible with reciprocity, and therefore no defeated profile λ can beat μ^A on the i^{th} coordinate . If on the other hand *i* is a shared M-coordinate, then any defeated profile in D(A) cannot be lower than the shared $\alpha(i)$ value. This would in fact once again violate reciprocity because no other member of A could better λ on *i*. Since μ^A shares the same minimal value, no defeated λ can beat μ^A on M-coordinates either, and therefore λ is simply bounded by μ^A.

The theorem also shows the reverse property: that any profile λ bounded by μ^A is defeated by A as well. This follows because A always satisfies reciprocity relative to μ^A, because the minimum can beat a member of A only on R-coordinates, but on these coordinates some α_{min} is always guaranteed to beat it. The same holds for any other λ in $D(\mu^A)$, since by hypothesis they are simply bounded by μ^A. Therefore λ is defeated by A and thus belongs to D(A).

The theorem must however distinguish the cases where A is a singleton containing a unique profile α from the others, because in the singleton case μ^A is identical to α itself, and thus should not be part of D(A), whereas in all other cases μ^A is collectively defeated by the members of A and hence belongs to D(A).

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[2] The bounding minimum of a set A is collectively defeated but not always collectively bounded by A. The set A is in fact guaranteed to meet reciprocity relative to μ^A but not strictness. See appendix A for an example.

[3] From an order-theoretic perspective, profiles form a lattice under the coordinate-wise order, where $\alpha<\beta$ if and only if $\forall i$, $\alpha(i)\leq\beta(i)$ and for some j, $\alpha(j)<\beta(j)$. The bounding minimum μ^A can then be defined in terms of the *greatest lower bound* of A, or *meet* of A, expressed as '$\wedge$A'. In particular, $\mu^A(i)=\wedge A(i)$ on M-coordinates, and $\mu^A(i)=1+\wedge A(i)$ on all others.

(21) **Thm. Bounding Minimum**. Let A be any set of profiles in V, and let μ^A be its corresponding bounding minimum, and λ a profile in V. Then λ is defeated by A iff it is simply bounded by $\{\mu^A\}$ or, when $|A|>1$, coincides with it.

$$|A|=1 \quad D(A)=D(\mu^A)$$
$$|A|>1 \quad D(A)=D(\mu^A)\cup\{\mu^A\}.$$

Pf. Let A be a defeating set for λ and μ^A its bounding minimum.
If $A=\{\alpha\}$, then all coordinates are M-coordinates, therefore by definition of minimum $\mu^A=\alpha$. It follows $D(\mu^A)=D(\alpha)=D(A)$.
Let us now assume $|A|>1$.
- Let us first prove $D(A)\subseteq D(\mu^A)\cup\{\mu^A\}$.
1.Let *i* be an M-coordinate shared across A. Then $\forall\alpha\in A$, $\lambda(i)\geq\alpha(i)=\mu^A(i)$, else reciprocity would require the existence of some $\alpha'\in A$, such that $\alpha'(i)<\lambda(i)$, against the hypothesis that $i\in M^A$.
2. Let *i* be a non-shared R-coordinate, and let α_{min} be the member of A with the minimal value for *i*.
2.1. Then $\lambda(i)>\alpha_{min}(i)$, else since *i* is not shared there would necessarily exist some $\alpha\in A$ such that $\lambda(i)\leq\alpha_{min}(i)<\alpha(i)$, and then by reciprocity some $\alpha'\in A$, such that $\alpha'(i)<\lambda(i)<\alpha(i)$, contradicting the hypothesis that $\alpha_{min}(i)$ is minimal across A.
2.2. It follows that $\lambda(i)\geq\mu^A(i)$, because $\mu^A(i)=\alpha_{min}(i)+1$ is the lowest available value above $\alpha_{min}(i)$.
3. Since M^A and R^A partition the set of coordinates, it follows that $\forall i$, $\lambda(i)\geq\mu^A(i)$.
4. Either $\lambda=\mu^A$, or $\exists j$, $\lambda(j)>\mu^A(j)$ and therefore $\lambda\in D(\mu^A)$, proving $D(A)\subseteq D(\mu^A)\cup\{\mu^A\}$.
- Let us now show $D(\mu^A)\cup\{\mu^A\}\subseteq D(A)$.
1. Let $\lambda\in D(\mu^A)\cup\{\mu^A\}$, then by the defeating singleton lemma, $\forall i$ $\lambda(i)\geq\mu^A(i)$.
2. Let *i* be an M-coordinate shared across A. Then, by the definitions of μ^A and of M-coordinate and by point 1, $\forall\alpha\in A$ $\alpha(i)\leq\lambda(i)$, and therefore reciprocity is vacuously satisfied on *i*.
3. Let *i* be a non-shared R-coordinate, and let α_{min} be the member of A with the minimal value for *i*.
3.1. Then, by 1 and by definition of μ^A, $\alpha_{min}(i)<\mu^A(i)\leq\lambda(i)$ and therefore reciprocity is satisfied on *i* because α_{min} will rescue any profile α where $\lambda(i)<\alpha(i)$.
4. Since no other coordinates are given, and A satisfies reciprocity on λ, it follows $\lambda\in D(A)$, and hence $D(\mu^A)\cup\{\mu^A\}\subseteq D(A)$. □

As a concrete example, consider a three-dimensional case where a profile $\lambda=\langle 1,4,5\rangle$ sharing with α and β the C_3-value is nevertheless collectively bounded by $\alpha=\langle 0,7,5\rangle$ and $\beta=\langle 4,2,5\rangle$ via reciprocity on C_1 and C_2, with α beating λ on C_1 and β on C_2. The minimum $\mu^A=\langle 1,3,5\rangle$ simply bounds λ, because it beats it on C_2 and equals it on C_1 and C_3. Furthermore no arbitrary defeated profile λ can beat the minimum on any coordinate. For example, λ cannot be lower than 5 on C_3, because then reciprocity would require some member of A to have a value lower than 5 on C_3, but A contains no such profile. For the same reason, λ cannot be lower than 1 on C_1, since reciprocity would again require some member of A to beat λ on this coordinate, and none is available. The arbitrary bounded profile λ thus has as lower bounds on each coordinate precisely the coordinate values assigned to μ^A, which therefore simply bounds it.

4. Reducing Defeating to Simple Bounding

We now proceed to our first main goal, i.e. characterizing L(A) in terms of simple bounding alone. This result is straightforward as an abstract property of profile sets, while it gives rise to some interesting issues and solutions when we try to use it to provide an algorithm determining L(A).

4.1 Subset Driven Reduction

The loser set L(A) can be completely computed in terms of simple bounding by the relevant bounding minima. By the bounding theorem, every loser in L(A) must be bounded by some subset B of A, and since every bounding set is also a defeating set, it must be simply bounded by the corresponding minimum μ^B, or possibly coincide with it when B is not a singleton. In either case, the loser is in D(B), and the union of all D(B) associated with all possible subsets of A will match L(A). The theorem and its proof are provided below.

(22) **Thm. L-Decomposition**. Let A be a profile set in the space V determined by the constraint coordinate set Σ and λ a profile in L(A), then λ is either defeated by or identical to some bounding minimum μ^B for some B in A.

$$L(A) = \cup_{B\subseteq A,\, B\neq\emptyset} D(B) = [\cup_{B\subseteq A} D(\mu^B)] \cup [\cup_{B\subseteq A,\, |B|>1} \{\mu^B\}].$$

Pf. Let A be any set in V, and λ a profile in L(A). By the loser set lemma $L(A) = \cup_{B\subseteq A,\, B\neq\emptyset} D(B)$. The reduction to the expression '$[\cup_{B\subseteq A} D(\mu^B)] \cup [\cup_{B\subseteq A,\, |B|>1} \{\mu^B\}]$' then follows by replacing each D(B) with its equivalent decomposition spelled out in the bounding minimum theorem. □

The above theorem presents two important properties and one drawback.

The most important property is this: whether a profile is turned into a loser by a set A reduces to testing whether it is simply bounded by some bounding minimum, the coordinates of which we can derive as soon as we know A. The related drawback is that each bounding minimum is associated to a *subset* of A, and thus L(A) in principle requires computing $2^{|A|}$ minima. As we will see in the next section, this inherent complexity can be brought down considerably once other properties of minima are exploited.

The second positive property of the above reduction is its independence from the winner/loser status of the members of A, which was always left undetermined and unconstrained. In particular, the set could contain profiles which are themselves losers in A, i.e. defeated by some proper subset of A, and yet the above theorem would still include them in L(A), while properly excluding all those profiles that are winners in A, i.e. cannot be defeated by any of its subsets.

Consider for example the set $A=\{\alpha, \beta, \lambda_1, \lambda_2\}$ with $\alpha=\langle 2,6\rangle$, $\beta=\langle 6,2\rangle$, $\lambda_1=\mu^A\langle 3,3\rangle$, and $\lambda_2=\langle 2,7\rangle$. Note that the profiles λ_1 and λ_2 are defeated by the subset $B=\{\alpha, \beta\}$. Let us now compute L(A) through the formula in the theorem on L-decomposition. Profile λ_1 would be included because it coincides with μ^A, which is part of D(A), and therefore L(A), whenever A is not a singleton. Likewise, λ_2 would enter L(A) because it is simply bounded by α, and thus it is included in $D(\mu^B)$

when B={α}, in which case μ^B coincides with α itself. The profiles α and β, on the other hand, would never enter L(A). They in fact are not bounded by any μ^B, and they coincide with a minimum only when they form their own singleton, but minima are not part of L(A) when B is a singleton, see the bounding minimum theorem. The formula thus properly discriminates between losers and winners within the original set A. This result is recorded in the corollary below.

(23) **Corollary. Losers in A**. Let A be a profile set in the space V, and λ a profile in A, then λ belongs to L(A) iff λ is defeated by some non-empty subset of A.

$$\forall\lambda\in A,\ [\ \lambda\in L(A) \Leftrightarrow \exists A'\subseteq A,\ A'\neq\emptyset,\ A'\sqsubset\lambda\ ].$$

Pf. (⇒) Let S=A−{λ}, then by the bounding theorem $\exists A'\subseteq S\subseteq A$, $A'\neq\emptyset$, and A′ a bounding set for {λ}. Then by the defeating bounding lemma A′ is a defeating set for λ , hence $A'\sqsubset\lambda$.

(⇐) Let λ∈A and $\exists A'\subseteq A$, $A'\neq\emptyset$, $A'\sqsubset\lambda$. Then by the defeating bounding lemma A′ includes some non-empty set B constituting a bounding set for λ, and since $B\subseteq A'\subseteq A$ it follows λ∈L(A). □

4.2 Complexity of Bounding Reduction

How many bounding minima are needed to completely characterize the loser set L(A) in terms of simple bounding alone? Excluding the null set, a set A contains $2^{|A|}-1$ subsets, each determining its corresponding minimum. While this is a finite figure, and hence better than attempting to list the infinite set of losers one by one, it grows exponentially with the size of A, yielding, for example, 1,048,575 minima for a simple set of 20 profiles.

Most of these minima, however, are copies or simply bound each other, and hence are not really necessary. Consider for example the set $A=\{\langle 1,1,2\rangle_\alpha, \langle 3,6,1\rangle_\beta, \langle 6,3,1\rangle_\gamma\}$. Although these are all winner profiles, the minimum for B={α, β} is identical to that for B′={α, γ}, namely $\mu=\langle 2,2,2\rangle$, which is also the bounding minimum for the superset A. The reason is that in both cases, α determines the first and second coordinates of the three minima, because it is minimal in A for these two coordinates, while β and γ only determine the value for the third one. on which they coincide.

Obviously, we are only interested in the minima which are necessary to compute L(A). We may begin to exclude unnecessary minima by noticing that for any set A, the only subsets worth considering are those whose minimum is not already bounded by the minimum of A, i.e. those whose minimum beats the minimum for the entire A on at least one coordinate. The following theorem shows that this may occur if and only if on some R-coordinate of A all members of the subset coincide with the minimal value in A for that coordinate, giving us a tool to identify the relevant subsets.

(24) **Thm. Relevant Minima**. Let A be a profile set in the space V, and B a subset of A. Then μ^B is not harmonically bounded by μ^A iff there is a coordinate *i* in R^A (hence not shared by the members of A) which is also in M^B (hence shared by all members of B) such that for any member β of B, $\beta(i)$ is minimal in A, (and hence $\mu^B(i)=\beta(i)=\alpha_{min}(i)$).

$$\forall A,B\in V,\ B\subseteq A\ \ [\mu^B\notin D(\mu^A)\cup\{\mu^A\} \Leftrightarrow \exists i,\ i\in R^A,\ i\in M^B,\ \forall\beta\in B,\ \beta(i)=\alpha_{min}(i)\ ]$$

Pf. ($\Rightarrow$) By hypothesis: $\forall A,B\in V,\ B\subseteq A\ \ \mu^B\notin D(\mu^A)\cup\{\mu^A\}$. By the defeating singleton lemma: $\exists i,\ \mu^B(i)<\mu^A(i)$.

1. Then $i\in R^A$. Assume this were not the case and $i\in M^A$. Then by definition of minimum and M^A it follows that

$$\forall\alpha\in A,\ \exists\beta\in B,\ \beta(i)\leq\mu^B(i)<\mu^A(i)=\alpha(i),$$

hence $\forall\alpha\in A,\ \exists\beta\in B,\ \beta(i)<\alpha(i)$, which is contradictory because by definition of M^A $\beta(i)=\alpha(i)$.

2. Moreover, $i\in M^B$. Assume this were not the case and $i\in R^B$. Then by definition of minimum:

$$\exists\alpha_{min}\in A,\ \exists\beta_{min}\in B,\ \beta_{min}(i)+1=\mu^B(i)<\mu^A(i)=\alpha_{min}(i)+1.$$

It follows that

$$\exists\alpha_{min}\in A,\ \exists\beta_{min}\in B,\ \beta_{min}(i)<\alpha_{min}(i),$$

which is contradictory because $B\subseteq A$, and hence α_{min} is minimal on *i* in B as well.

3. Since $i\in M^B$, and $i\in R^A$, by definition of minimum and of M- and R-coordinates, it follows for *i* that

$$\exists\alpha_{min}\in A,\ \forall\beta\in B,\ \beta(i)=\mu^B(i)<\mu^A(i)=\alpha_{min}(i)+1,$$

which holds if and only if $\forall\beta\in B,\ \beta(i)=\alpha_{min}(i)$ because α_{min} is minimal on *i* in B as well.

($\Leftarrow$) By hypothesis,

$$\forall A,B\in V,\ B\subseteq A,\ \exists i,\ i\in R^A,\ i\in M^B,\ \forall\beta\in B,\ \beta(i)=\alpha_{min}(i).$$

By definition of minimum and of M- and R- coordinate it follows that

$$\exists i,\ \exists\alpha_{min}\in A,\ \forall\beta\in B,\ \mu^B(i)=\beta(i)<\alpha_{min}(i)+1=\mu^A(i)$$

and therefore that $\exists i,\ \mu^B(i)<\mu^A(i)$ and hence $\mu^B\notin D(\mu^A)\cup\{\mu^A\}$. □

A useful corollary follows by replacing the minima with the corresponding subsets, showing that the only relevant subsets for defeating those losers not yet defeated by A are those that share among their members some coordinate value minimal across A. Intuitively, if a subset B defeats some more additional losers than A, then its minimum cannot be defeated by the minimum of A, else by transitivity of bounding anything bounded by it would also be bounded by the minimum for A. This condition meets the condition for the relevant minima theorem above, forcing B to share some minimal coordinate value from A.

The reverse entailment holds too, but only in multidimensional violation spaces, telling us that any subset sharing some minimal value of its superset will defeat some additional profiles not defeated by the superset. The demonstration is trivial, as any subset B with minimal value *min* in A for some dimension *i* will defeat any profile equally minimal on *i* and non-minimal on any other coordinate. This profile is not defeated by A, whose bounding minimum is necessarily equal to *min+1* on coordinate *i*.

One-dimensional spaces, on the other hand, allow for degenerate cases where one member α of the superset bounds the other members. The subset B then collects α as its unique element. As a consequence, the minimum μ^A is identical to μ^B except on *i*, where it holds that $\mu^A(i)=\mu^B(i)+1$. The minimum μ^B thus remains undefeated by μ^A, as required by the theorem on relevant minima, but D(B) becomes identical to D(A) because of the different definitions of D(S) for singleton and non-singleton sets. For example, let $A=\{\langle 1\rangle, \langle 3\rangle\}$ and $B=\{\langle 1\rangle\}$. By the bounding minimum theorem the set D(A) is equivalent to $D(\mu^A)\cup\{\mu^A\}$, and since $\mu^A=2$ it includes any $\lambda\geq 2$. Since B is a singleton, D(B) is equal to $D(\mu^B\}$ alone, and since $\mu^B=1$, it includes all profiles $\lambda>1$. The two sets thus coincide, and hence B adds no new defeated profiles. The relevant minima theorem however remains valid, because μ^B is not in $D(\mu^A)$.

(25) **Corollary. Relevant Subsets**. Let A be a profile set in the space V, and B a subset of A. Then the set of profiles defeated by B is not a subset of those defeated by A only if there is a coordinate *i* in R^A and M^B such that for any member β of B, $\beta(i)$ is minimal in A. When Σ has two or more coordinates, the reverse entailment holds as well.

$$(\Rightarrow)\quad \forall A,B\in V,\ B\subseteq A\ \ [\ D(B)\not\subseteq D(A) \Rightarrow\ \exists i,\ i\in R^A,\ i\in M^B,\ \forall\beta\in B,\ \beta(i)=\alpha_{min}(i)\ ]$$

$$(\Leftarrow)\quad \forall\Sigma,\ |\Sigma|=1,\ \forall A,B\in V,\ B\subseteq A\ \ [\ D(B)\not\subseteq D(A) \Leftarrow\ \exists i,\ i\in R^A,\ i\in M^B,\ \forall\beta\in B,\ \beta(i)=\alpha_{min}(i)\ ]$$

Pf. ($\Rightarrow$) By hypothesis, $\exists\ \lambda\in D(B)$ and $\lambda\notin D(A)$. Hence $B\neq A$, and since $A\subseteq B$, also $|A|>1$.

1. By the bounding minimum theorem applied to sets A and B, $\lambda\in D(\mu^B)$ and $\lambda\notin D(\mu^A)\cup\{\mu^A\}$.
2. It follows that $\mu^B\notin D(\mu^A)\cup\{\mu^A\}$, because otherwise by bounding transitivity $D(\mu^B)\subseteq D(\mu^A)$, and therefore $\lambda\in D(\mu^A)$ contrary to 1.
3. From $\mu^B\notin D(\mu^A)\cup\{\mu^A$, by relevant minima, it follows that $\exists i,\ i\in R^A,\ i\in M^B,\ \forall\beta\in B,\ \beta(i)=\alpha_{min}(i)$.

($\Leftarrow$) By hypothesis, $\exists i,\ i\in R^A,\ i\in M^B,\ \forall\beta\in B,\ \beta(i)=\alpha_{min}(i)$. Let $B=\{\alpha_{min}\}$.

1. By definition of bounding minimum it holds that $\mu^A(i)=\alpha_{min}(i)+1$ and $\mu^B(i)=\alpha_{min}(i)$.
2. Consider any profile λ such that $\lambda(i)=\alpha_{min}(i)$ and $\forall j\neq i\ \lambda(j)>\alpha_{min}(j)$. Obviously $\lambda\in D(B)$. Note that *j* exists, because by hypothesis $|\Sigma|>1$, and therefore λ exists too.
3. Since $\alpha_{min}(i)=\lambda(i)<\mu^A(i)=\alpha_{min}(i)+1$, it follows that $\lambda\notin D(\mu^A)\cup\{\mu^A\}=D(A)$, and therefore $\lambda\notin D(A)$. □

The theorem and its corollary provide a powerful tool to examine how many minima are necessary to compute L(A). Since only subsets sharing some minimal coordinate matter, we may construct them in a systematic fashion, finding for every coordinate *i* the minimal value $\alpha_{min}(i)$ in A and then checking the subsets sharing this minimal value across its members. Consider again the case where $A=\{\langle 1,1,3\rangle_\alpha, \langle 3,6,1\rangle_\beta, \langle 6,3,1\rangle_\gamma\}$ with $\mu^A=\langle 2,2,2\rangle$. According to the corollary, the sets $B_1=\{\langle 1,1,3\rangle_\alpha, \langle 3,6,1\rangle_\beta\}$ and $B_2=\{\alpha\ \langle 1,1,3\rangle_\alpha, \langle 6,3,1\rangle_\gamma\}$ are irrelevant, because their members do not share any minimal value for A. These are indeed two sets whose bounding minima μ^{B1} and μ^{B2} coincide with the minimum $\mu^A=\langle 2,2,2\rangle$, and hence cannot defeat any additional profiles. The only relevant non-singleton set is $B=\{\langle 3,6,1\rangle_\beta, \langle 6,3,1\rangle_\gamma\}$, which shares the minimal value '1' on the third coordinate C_3. The corresponding minimum, $\mu^B=\langle 4,4,1\rangle$ is not bounded by μ^A and adds to L(A) all the profiles collectively bounded by β and γ that share their minimal value on C_3, and hence are not defeated by A because reciprocity could not be satisfied on C_3.

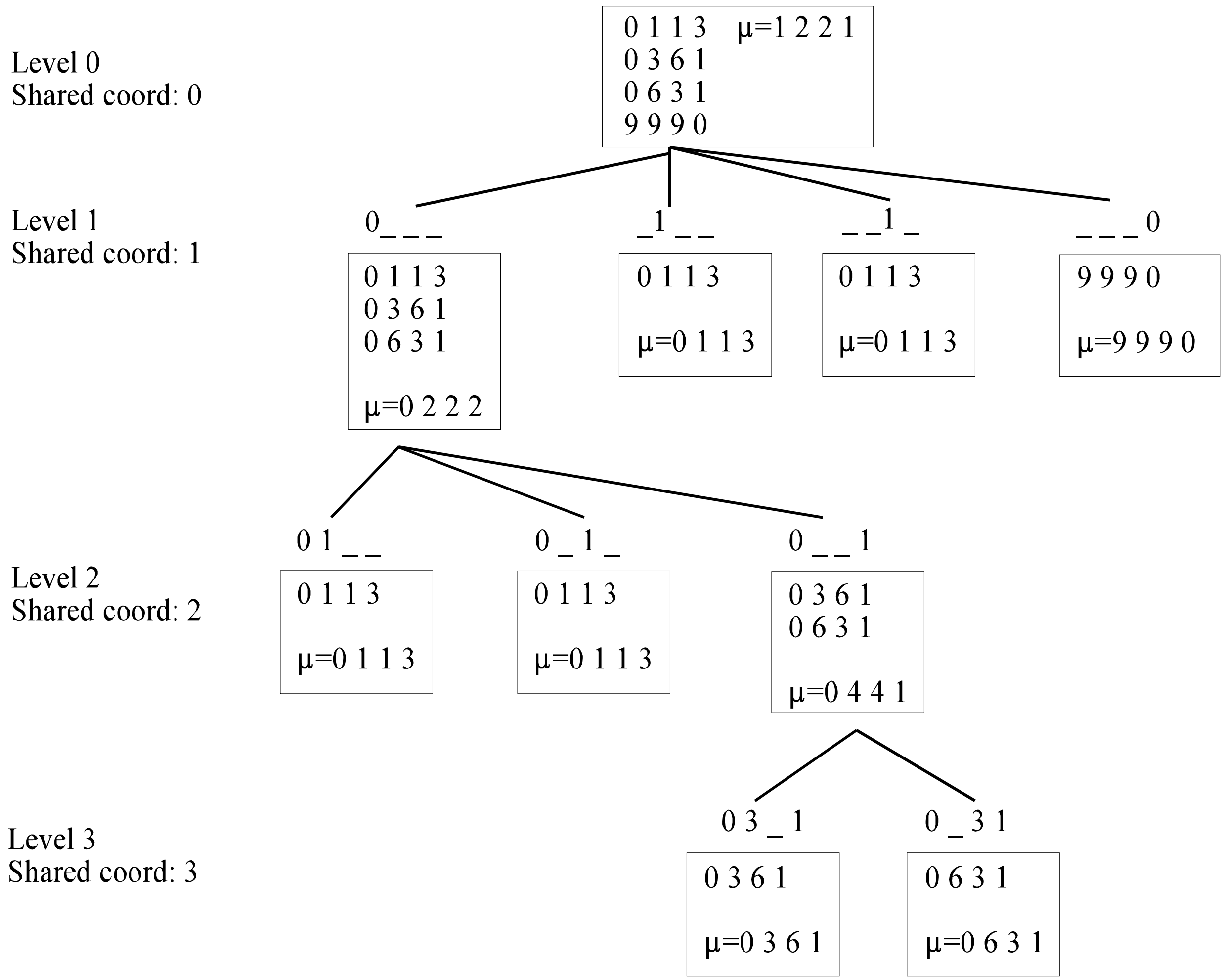
Singletons can be relevant subsets as well. Profile $\alpha=\langle 1,1,3\rangle$, for example, posts the minimal values for C_1 and C_2, and thus forms a relevant subset. Its relevance is easily assessed once one considers that α simply bounds infinitely many losers of the kind $\lambda=\langle 1, 1+i, 3+k\rangle$, with $i,k\geq 0$, none of which are bounded by $\mu^A=\langle 2,2,2\rangle$ which is too high on C_1.

The corollary also has an important recursive aspect: it can be applied again to each relevant subset. For example, let $A=\{\langle 0,1,1,3\rangle_\alpha, \langle 0,3,6,1\rangle_\beta, \langle 0,6,3,1\rangle_\gamma, \langle 9,9,9,0\rangle_\delta\}$. A first relevant subset is $B=\{\langle 0,1,1,3\rangle_\alpha, \langle 0,3,6,1\rangle_\beta, \langle 0,6,3,1\rangle_\gamma\}$ which is minimal on C_1. The minimum $\mu^B=\langle 0,2,2,2\rangle$ ensures that losers sharing the same minimal value on C_1 will be bounded. We may now apply the corollary again to this subset, and notice that within B, the subset $C=\{\langle 0,3,6,1\rangle_\beta, \langle 0,6,3,1\rangle_\gamma\}$ forms an additional relevant subset sharing two minimal coordinate values for the superset B, namely C_1 and C_4, and yielding the minimum $\mu^C=\langle 0,4,4,1\rangle$, which will collectively bound those losers posting the same minimal values on C_1 and C_4. For example, the loser $\lambda=\langle 0,5,5,1\rangle$ is bounded by μ^C, but neither by $\mu^B=\langle 0,2,2,2\rangle$ nor $\mu^A=\langle 1,2,2,1\rangle$.

It follows that we may systematically seek relevant subsets by applying the corollary recursively. Let A be a set of profile in V^N. At level zero, we build all the N largest subsets sharing the first, or second, or third, ..., or n^{th} coordinate value minimal in A. For each discovered subset S, we repeat the procedure fixing one of the $n-1$ coordinates not yet fixed, creating further relevant subsets. The procedure applies recursively until all coordinates are fixed and all subsets are singletons formed by the single profiles. By the above theorem and corollary, every new subset so discovered is relevant, adding new defeated profiles to L(A) via simple bounding by the corresponding bounding minimum. Consider for example once again $A=\{\langle 0,1,1,3\rangle_\alpha, \langle 0,3,6,1\rangle_\beta, \langle 0,6,3,1\rangle_\gamma, \langle 9,9,9,0\rangle_\delta\}$. The computation of the subsets proceeds as shown in the figure below. When a relevant subset contains only one profile, no further branching occurs, because all coordinates of a singleton are shared, albeit only vacuously so, therefore they are not R-coordinates and no new relevant subset can be built.

The tree starts at level 0, where no coordinate is yet fixed; this is the root node of the tree, and the associated subset is the entire A, with $\mu^A=\langle 1,2,2,1\rangle$. The maximal subsets sharing one minimal coordinate yield the four subsets shown at level 1. The only non-singleton subset occurs when minimizing C_1, allowing for further subset searches. All other subsets contain only one profile: the corresponding μ is identical to the candidate, and no further search is conducted. Note that the same profile may occur in multiple branches of the tree: this happens whenever a profile contains more than one minimal coordinate for some of its supersets. For example, $\alpha=\langle 0,1,1,3\rangle$ is minimal on C_1, C_2, and C_3 in A, and again on C_2 and C_3 on the leftmost level-1 subset.

(26) Recursive tree of relevant subsets:



In the following, we formalize the procedure for building relevant subsets by first defining the tree of relevant subsets T(A) for a profile set A, and then demonstrating how the sets in T(A) are sufficient to defeat any loser in L(A). We will then use the tree to establish an upper bound on the number of minima required to compute L(A). A priori, this would seem to be as high as the number of available subsets, i.e. 2^K-1 for any set of size *K*. But not all subsets are relevant and when this factor is taken into account the final number is considerably lower. For example, the example in the above figure requires only seven distinct subsets —and hence only seven minima— to simply bound any loser in L(A), rather than the fifteen suggested by the number of possible non-empty subsets .

The tree of relevant subsets T(A) for a profile set A is defined recursively level by level starting from level 0. First we defined a minimal-coordinate subset relative to some profile set A and coordinate *c* as the subset ‘c(A)’ of profiles posting the minimal value available in A for coordinate *c*. This definition permits us to track which coordinates get recursively fixed. For example, in the above case, the four subsets for level 1 would be characterized as *1(A)*, *2(A)*, *3(A)*, and *4(A)*, as in figure (30) below. The first subset shown for level 2 would be *2(1(A)*. This also records at once for each subset which coordinates have already been fixed and in what order. The order is crucial, because each relevant subset is only sensitive to the minimal coordinates of its immediate superset. For example, *4(1(A))* fixes as minimal for the fourth coordinate the value '1', i.e. the minimal value available in *1(A)*, rather than the value '0' available on A. In contrast, *1(4(A))*, with the reverse order, does not correspond to any relevant set, because *4(A)* fixes the 4^{th} coordinate to '0' forming the singleton subset containing $\langle 9,9,9,0\rangle$.

(27) **Def. Min-coordinate Subset**. Let A be a profile set, and *c* a coordinate in the set of coordinates Σ, then the subset 'c(A)' with minimal coordinate *c* relative to A is defined as the set of profiles in A with minimal c-coordinate across A.

$$c(A)=\{\alpha:\alpha\in A,\ \forall\alpha'\in A,\ \alpha(c)\leq\alpha'(c)\}$$

As an auxiliary tool, we define the set *Fixed*(A) which returns the set of coordinates that have already been fixed for some minimal-coordinate subset A. This will help us prevent fixing again a coordinate that has already been fixed.

(28) **Def. Fixed Coordinates**. Let B be a recursively determined min-coordinate set of the form $B=c_1(..(c_k(A))$, then the set Fixed of fixed coordinates for B includes all the fixed coordinates c_i determining B.

$$B=c_1(..(c_k(A))) \Rightarrow \text{Fixed}(B)=\{c_1,..,c_k\}$$

We may now introduce the definition of T(A), which collects together all the relevant subsets determined by each level of the tree. The first level, L_0, contains the initial set of profiles A, where no minimal coordinate value has been fixed yet. Then, any successive level L_i is defined in terms of the immediately precedent level L_{i-1} as collecting any new subset that can be formed from those in L_{i-1} by fixing a minimal value among one not yet minimal R-coordinate. Note that as soon as a singleton is formed, no further subsets are sought, because all coordinates become shared M-coordinates.

(29) **Def. T(A)**. Let Σ the set of constraints determining the coordinates of V, and A a set of profiles in V. Then the tree of relevant subsets is built level by level according to the following recursive steps, with each level L_i collecting all the relevant subsets determined on the basis of the immediate preceding level. The tree of relevant minimal subsets T(A) for A is then defined as the union of all levels.

$T(A) = \cup_i L_i$, where each L_i is defined as follows:

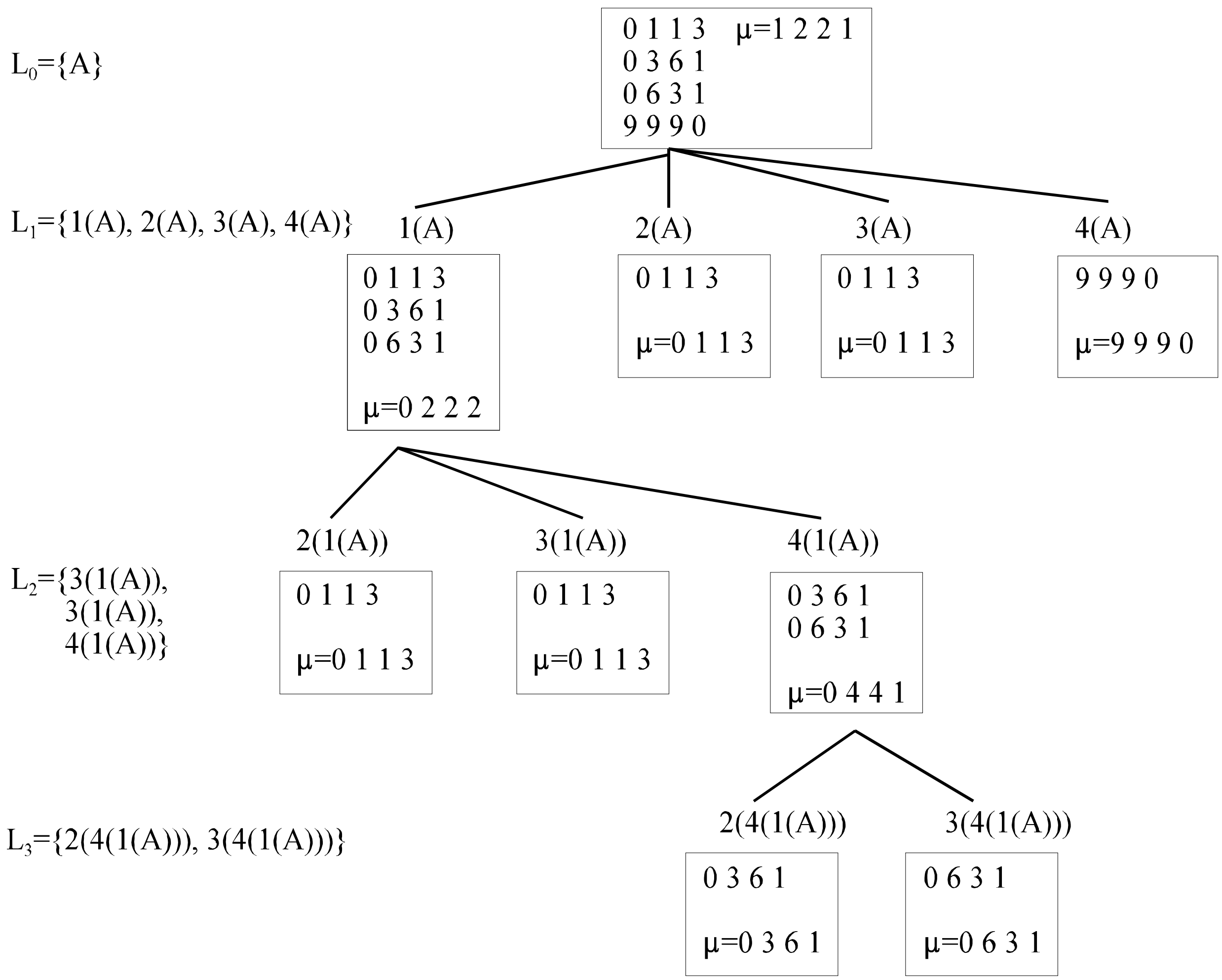
Step 0: $L_0=\{A\}$.

Step i: $L_i=\{B : B=c(B'),\ B'\in L_{i-1},\ c\in R^{B'},\ c\in\Sigma,\ c\notin \text{Fixed}(B')\}$.

The example considered above is reexamined in the figure below with the corresponding subsets and levels used in the above definitions.

(30) Recursive tree of relevant subsets:

T(A)={A, 1(A), 2(A), 3(A), 4(A), 3(1(A)), 3(1(A)), 4(1(A)), 2(4(1(A))), 3(4(1(A)))}



We now need to demonstrate that the relevant subsets in T(A) are sufficient to compute L(A). That each profile defeated by some set in T(A) is a loser in L(A) follows trivially from the L-decomposition theorem (22) on p. 15, because L(A) collects any profile defeated by any subset of A. Any loser in L(A) is defeated by some subset in T(A): this follows from the relevant subsets corollary in (25) above applied recursively to the sets in T(A): by L-decomposition, each loser is defeated by some subset B, and by the corollary the members of B must share some value minimal in A. The subsets of level L_1 within T(A) are the largest possible subsets satisfying this condition, therefore B is either one of them, proving the theorem, or it is a subset of one of them. In this latter case the same reasoning applies again, level by level in a recursive fashion. We are guaranteed to discover B, because each application reduces the size of the sets being considered eventually reaching the singletons in T(A). For B to differ from all sets in T(A) is not possible, as the corollary on relevant minima forces it to be a subset of some set in T(A), and once the singletons in T(A) are reached the only additional possible subset is the empty set, but this contradicts L-decomposition, which requires B to be non-empty.

A possible cause of confusion arises from the fact that a loser in L(A) could be defeated by some set B′ not in T(A) whose members do not share any coordinate value. A loser, however, can be defeated by many distinct sets, and the theorem simply asserts that one of these sets is certainly in T(A), and this is the subset which the demonstration focuses on.

(31) **Thm. Sufficiency of T(A)**. Let A be a set of profiles and T(A) its corresponding tree of relevant subsets. Then a profile λ is a loser in L(A) if and only if it is defeated by some subset B in T(A) (i.e.iff it coincides or is simply bounded by μ^B forsome subset B in T(A)).

$$\lambda \in L(A) \Leftrightarrow \exists B \in T(A), B \sqsubset \lambda.$$

Pf. (⇐) By hypothesis, $\exists B \in T(A)$, $B \sqsubset \lambda$. By definition of T(A), $B \subseteq A$ and $B \neq \emptyset$. By L-decomposition, $\lambda \in L(A)$.

(⇒) Let $\lambda \in L(A)$ and let L_i be determined according to the definition for T(A).

1.By L-decomposition $\exists B \subseteq A$, $B \neq \emptyset$, $B \sqsubset \lambda$. Since $T(A) = \cup_i L_i$, all we need to show is $\exists i, B \in L_i$.

2. We assume this is false, i.e. $\nexists i, B \in L_i$, and derive a contradiction.

3. As we show in 3.1 and 3.2 below by induction on L_i, it holds: $\forall L_i, \exists B' \in L_{i+1}, B \subset B'$.

3.1 Consider $L_0 = \{A\}$. By hypothesis $A \not\sqsubset \lambda$, else B=A against 2. By the corollary on relevant subsets, B shares a minimal value $\alpha_{min}(i)$ on some coordinate *i* in R^A. Therefore, by definition of L_i, $\exists B' \in L_1, B \subseteq B'$, and since by 2 above $B \notin L_1$, it holds $\exists B' \in L_1, B \subset B'$, proving the property for *i*=0.

3.2 Let the property hold for L_i and let us derive it for L_{i+1}. Because it holds of L_i, it follows $\exists S \in L_i$, $B \subseteq S$ and also $S \not\sqsubset \lambda$, else B=S against 2. By the corollary on relevant subsets, B shares a minimal value $\sigma_{min}(i)$ for $\sigma_{min} \in S$ on some coordinate *i* in R^S. Therefore, by definition of L_i, $\exists B' \in L_{i+1}, B \subseteq B'$, and since by 2 above $B \notin L_{i+1,}$ it holds $\exists B' \in L_{i+1}, B \subset B'$.

4. Each subset S in L_{i+1} is a *proper* subset of some set S′ in L_i, because by definition of T(A) the coordinate whose value $\sigma_{min}(i)$ is shared across S must be in $R^{S'}$, and therefore $\exists \sigma \in S'$ $\sigma(i) > \sigma_{min}(i)$. It follows that $\forall S \in L_{i+1}$ $\exists S' \in L_i$ $|S| < |S'|$.

5. Let *max* be the highest level in T(A). By 4 level *max* is defined and finite because |A| is finite. Then by 3, $\exists B' \in L_{max+1}, B \subset B'$, contradicting the hypothesis that *max* is maximal. □

5. Preliminary Observations on the Complexity of Bounding Reduction

A bounding minimum is *necessary* when it is not defeated by some other set in T(A). Here we examine the following question: given a set of winner-profiles A of size *K*, how many *necessary* minima are there in the corresponding T(A)? For simplicity, we assume throughout that A consists of winners only. More general sets should be reducible to this case via simple bounding, and via their bounding minima, whose definition makes no similar assumption about the original set. Also recall in the following discussion that the members of A are themselves relevant bounding minima, and hence eventually contribute to the total number of minima.

The first section examines the conditions for necessary minima. The next considers how the number of dimensions relative to the size of a set affects the availability of necessary minima. The last one presents some interesting cases, either because they maximize coordinate sharing, or because they minimize it.

5.1 Conditions on Necessary Minima:

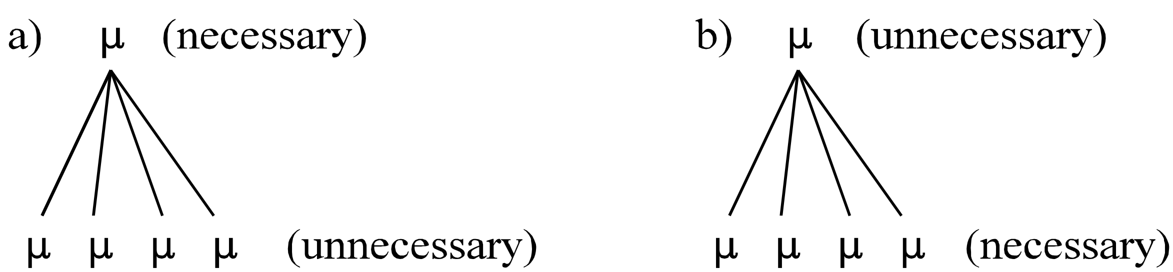
The bounding minimum for a set A is *necessary* only if required to defeat via simple bounding at least one profile not defeated by any of the relevant subsets of A, as in the definition below.

(32) **Def. Necessary Minima**. Let A be a set of profiles corresponding to some node in T(A), and B one of the relevant subsets for A in T(A). Then μ^A is necessary if and only if it is not defeated by any of its relevant subsets.

$$\mu^A \text{ is necessary} \Leftrightarrow \forall B\subseteq A,\ B\in T(A),\ B\not\subset\mu^A.$$

Not all minima of the relevant subsets composing T(A) are *necessary*, because some are simply bounded by others. The subset-superset relation between a node in T(A) and its daughters does not entail necessity. If we replace each node with the μ for the corresponding set, we can easily build examples where the μ of one node is necessary even though the μ's of all its daughters are bounded by some daughter of their own, as in case (a) below, or vice versa where the μ of a node is unnecessary even though the minima of all its daughters are necessary, as in (b) below.

(33) Necessary μ's.



An example of case (b) occurs for S={⟨8,2,2⟩, ⟨2,8,2⟩, ⟨2,2,8⟩}. The minimum μ^S=⟨3,3,3⟩ is bounded by the minimum of any pair of members, e.g. B={⟨2,8,2⟩, ⟨2,2,8⟩} with minimum μ^B=⟨2,3,3⟩, simple bounds μ^S. Note that μ^B is itself necessary, as it is not bounded by any of the members of B. The same is true for any other pair of members of S.

Case (a) is more complex, and has the above example as a component. Consider set A below. Each column groups together the profiles with the same minimal coordinate, thus identifying the 4 relevant subsets B_1=1(A), B_2=2(A), B_3=3(A), and B_4=4(A). Note how on any unshared coordinate each subset has the same structure as the set S from the previous example. The corresponding minima, given below, are thus bounded by their daughter subsets in T(A), each including two of their members. For example, $\mu^{B1}=\langle 0,3,3,3\rangle$ is bounded by $\mu^{C}=\langle 0,2,3,3\rangle$ corresponding to the subset $C=\{\langle 0,2,8,2\rangle, \langle 0,2,2,8\rangle\}$ of B1. Nevertheless, even if all minima for the subsets B_1-B_4 are bounded, the global minimum $\mu^{A}=\langle 1,1,1,1\rangle$ is not, nor is it defeated by any other minimum among A's relevant subsets.

(34)

$$\begin{array}{lllll} & B_1 & B_2 & B_3 & B_4 \\ A=\{ & \langle 0,8,2,2\rangle, & \langle 8,0,2,2\rangle, & \langle 8,2,0,2\rangle, & \langle 8,2,2,0\rangle, \\ & \langle 0,2,8,2\rangle, & \langle 2,0,8,2\rangle, & \langle 2,8,0,2\rangle, & \langle 2,8,2,0\rangle, \\ & \langle 0,2,2,8\rangle, & \langle 2,0,2,8\rangle, & \langle 2,2,0,8\rangle, & \langle 2,2,8,0\rangle \} \end{array}$$

$\mu^{A}=\langle 1,1,1,1\rangle$ $\quad\mu^{B1}=\langle 0,3,3,3\rangle$ $\quad\mu^{B2}=\langle 3,0,3,3\rangle$ $\quad\mu^{B3}=\langle 3,3,0,3\rangle$ $\quad\mu^{B4}=\langle 3,3,3,0\rangle$

The conditions determining whether the minimum of a profile set A is unnecessary because defeated by some relevant subset B, depends on the size of B. When B is not a singleton, it is sufficient that μ^{A} is lower than μ^{B} on some coordinate. This in turn translates into a condition on the minimal values available in A and B, according to the lemma below.

(35) **Lemma. Necessary Minima I**. Let A be a set of profiles corresponding to some node in T(A), and B a non-singleton relevant subset for A in T(A). Then μ^{A} is not defeated by B if and only if the following condition holds.

$$\forall B\subseteq A,\ B\in T(A),\ B\not\subset\mu^{A} \Leftrightarrow \begin{array}{l}[\ \exists i,\ i\in M^{B},\ \alpha_{min}(i)+1<\beta_{min}(i)\ ]\ \text{OR} \\ [\ \exists i,\ i\in R^{B},\ \alpha_{min}(i)<\beta_{min}(i)\ ].\end{array}$$

Pf. ($\Leftarrow$) Note that $i\notin M^{A}$, otherwise $\alpha_{min}(i)=\beta_{min}(i)$ contrary to hypothesis. Therefore, by definition of minimum, $\mu^{A}(i)=\alpha_{min}(i)+1$.

1. Assume only the first disjunct holds. Then $\mu^{B}(i)=\beta_{min}(i)$, and $\mu^{A}(i)=\alpha_{min}(i)+1<\beta_{min}(i)=\mu^{B}(i)$. Hence $B\not\subset\mu^{A}$.

2. Assume the second disjunct hold. Then $\mu^{B}(i)=\beta_{min}(i)+1$ and $\mu^{A}(i)=\alpha_{min}(i)+1<\beta_{min}(i)+1=\mu^{B}(i)$. It follows that $B\not\subset\mu^{A}$.

($\Rightarrow$) Since $|B|>1$, by hypothesis and by the bounding minimum theorem $\mu^{A}\notin D(\mu^{B})\cup\{\mu^{B}\}$.

1. By definition of defeating set for singletons, it follows $\exists i,\ \mu^{A}(i)<\mu^{B}(i)$.
2. Note that $i\notin M^{A}$, because in this case the minimal value $\alpha_{min}(i)$ is shared across both sets because B is a subset of A, yielding $\alpha_{min}(i)=\beta_{min}(i)$, and therefore $\mu^{A}(i)=\alpha_{min}(i)=\mu^{B}(i)$ against 1.
3. Therefore $i\in R^{A}$ and by definition of minimum $\mu^{A}(i)=\alpha_{min}(i)+1$.
4. If $i\in M^{B}$, then $\alpha_{min}(i)+1=\mu^{A}(i)<\mu^{B}(i)=\beta_{min}(i)$, and therefore $\alpha_{min}(i)+1<\beta_{min}(i)$.[4] Otherwise $i\in R^{B}$, and hence $\alpha_{min}(i)+1=\mu^{A}(i)<\mu^{B}(i)=\beta_{min}(i)+1$, and therefore $\alpha_{min}(i)<\beta_{min}(i)$. □

[4] An example where this condition applies is the following: $A=\{\langle 0,2,4,6\rangle, \langle 0,2,6,4\rangle, \langle 7,1,5,5\rangle\}$. Then $\mu^{A}=\langle 1,2,5,5\rangle$, bounded by $\mu^{B}=\langle 0,2,5,5\rangle$ for $B=\{\langle 0,2,4,6\rangle, \langle 0,2,6,4\rangle\}$. The minimal value for A is lower than that for B on the second coordinate, but $\alpha_{min}(i)+1$ is not, making bounding possible.

When the relevant subset B is a singleton, D(B) no longer includes μ^B. In this case, μ^A is a necessary minimum even if it coincides with μ^B, because μ^A is defeated by A, and thus crucially adds to L(A) a profile which would otherwise incorrectly escape loser status. Note that in singleton sets, there only are M-coordinates, and hence the second condition in the above lemma cannot apply.

(36) **Lemma. Necessary Minima II**. Let A be a set of profiles corresponding to some node in T(A), and B a singleton relevant subset for A in T(A). Then μ^A is not defeated by B if and only if the following condition holds.

$$\forall B\subseteq A,\ B\in T(A),\ B\not\subset\mu^A \Leftrightarrow \begin{array}{l} [\ \exists i,\ i\in R^A,\ \alpha_{min}(i)+1<\beta_{min}(i)\ ]\ \text{OR} \\ [\ \forall i,\ i\in R^A,\ \alpha_{min}(i)+1=\beta_{min}(i)\ ]. \end{array}$$

Pf. By hypothesis on B, $\forall i$, $\mu^B(i)=\beta_{min}(i)$.

($\Leftarrow$) Assume only the first disjunct holds. Then $\mu^A(i)=\alpha_{min}(i)+1$. Therefore $\mu^A(i)<\mu^B(i)$. Hence $B\not\subset\mu^A$. Assume the second disjunct holds. Then $\mu^A(i)=\alpha_{min}(i)+1=\beta_{min}(i)=\mu^B(i)$. Therefore $\mu^A=\mu^B$. Hence $B\not\subset\mu^A$, because by the bounding minimum theorem singletons do not defeat their own minima.

($\Rightarrow$) By hypothesis and by the bounding minimum theorem, $\mu^A\notin D(\mu^B)$.

1. By definition of defeating set for singletons, either $\exists i$, $\mu^A(i)<\mu^B(i)$ or $\forall i$, $\mu^A(i)=\mu^B(i)$.
2. Assume the first disjunct from 1 holds, i.e. $\exists i$, $\mu^A(i)<\mu^B(i)$. Then $i\notin M^A$, else $\mu^A(i)=\alpha_{min}(i)=\beta_{min}(i)=\mu^B(i)$ against hypothesis. Therefore $i\in R^A$. Hence $\mu^A(i)=\alpha_{min}(i)+1<\beta_{min}(i)=\mu^B(i)$.
3. Now assume the second disjunct from 1 holds, i.e. $\forall i$, $\mu^A(i)=\mu^B(i)$. Then, $\forall i$, $i\in R^A$, $\alpha_{min}(i)+1=\beta_{min}(i)$, and by definition of minimum, $\forall i$, $i\in R^A$, $\mu^A(i)=\alpha_{min}(i)+1=\beta_{min}(i)=\mu^B(i)$. □

5.2 Dimensions of the Violation Space

Can it happen that the number of dimensions of the space V^N affect the number of necessary minima needed for a profile set A of size K? While we do not yet have an exhaustive answer for this question, we show that the worst case scenario, where all 2^K-1 non-empty subsets of A constitute subsets with necessary minima, is impossible whenever $K\geq N$, while it becomes possible at least in some cases if $K<N$. The three possible relations of K to N are examined here below.

♦ **K>N**

Under these circumstances, the worst scenario is not possible..The result follows from a simple calculation over the number of relevant sets in T(A) required to accommodate the 2^K-1 subsets of A. The tree T(A) may at most identify N relevant subsets of size $K-1$, one for each dimension (see the definition of T(A) on p. 21 above). However, there are K subsets of size $K-1$ in A. Therefore, there simply are not enough relevant sets of the appropriate size in T(A) for the possible sets of the same size in the power set P(A). It follows that one or more sets in P(A) are necessarily irrelevant, since we know from the theorem on the sufficiency of T(A), on page 23, that its subsets and corresponding minima are sufficient to determine all losers in L(A).

♦ **K=N**

If K=N, the worst case scenario is equally impossible. Forming K subsets of size K−1 requires each member of A to occur in K−1 of these subsets. This entails that on each coordinate, K−1 members of A share the minimal value for A, else they could not be part of the corresponding relevant sets. This must hold on all K distinct coordinates, yielding K distinct subsets. But this is possible only if the original set A has the shape shown below, where each m_i is the global minimal value on the i^{th} coordinate, and each V_j is non-minimal.

$$
\text{(37)}\quad A=\left\{\begin{array}{l}
\langle\; m_1,\; m_2,\; m_3,\; ...,\; m_{k-1}, V_k \;\rangle_1,\\
\langle\; m_1,\; m_2,\; m_3,\; ...,\; V_{k-1}, m_k \;\rangle_2,\\
\langle\; m_1,\; m_2,\; m_3,\; ...,\; m_{k-1}, m_k \;\rangle_3,\\
\quad ....\qquad\qquad ...\qquad ....\\
\langle\; m_1,\; m_2,\; V_3,\; ...,\; m_{k-1}, m_k \;\rangle_{k-2},\\
\langle\; m_1,\; V_2,\; m_3,\; ...,\; m_{k-1}, m_k \;\rangle_{k-1},\\
\langle\; V_1,\; m_2,\; m_3,\; ...,\; m_{k-1}, m_k \;\rangle_k
\end{array}\right\}
$$

The minimum for A will then be $\mu^A=\langle m_1+1, m_2+1, m_3+1, ..., m_{k-1}+1, m_k+1\rangle$, and will be defeated by each of the K subset at issue here. The subset for coordinate *i* will in fact shares with μ^A all but the minimal value m_i, and thus bound μ^A.

For example, the minimum for 1(A) is $\mu^{1(A)}=\langle m_1, m_2+1, m_3+1, ..., m_{k-1}+1, m_k+1\rangle$, and bounds μ^A. At least one μ, that for A, is thus unnecessary, showing that the worst scenario is not possible for K=N.

♦ **K<N.**

The worst case scenario becomes possible when K<N. What has been said about the layout for the K=N case, remains true here too. Therefore there are K coordinates on each of which K−1 members of A share a minimal value for A. However, the remaining coordinates may host non-minimal values that ensure that the corresponding minima do not bound each other nor the global minimum for A. An example follows below for N=8 and K=4. To ease comparisons, global minimal values are underlined in the minima for non-singleton relevant subsets, and high values that prevent the minimum to bound other minima are bolded. The $2^4-1=15$ minima never bound each other.

(38) Worst Case: A={ α, β, γ, δ}

α	=	1	3	5	**9**	8	8	8	5	=	μ(α)
β	=	1	3	**9**	7	7	7	5	7	=	μ(β)
γ	=	1	**9**	5	7	9	5	7	8	=	μ(γ)
δ	=	**9**	3	5	7	5	9	9	9	=	μ(δ)
		2	4	6	8	6	6	6	6	=	μ^A
		$\underline{1}$	4	6	8	**8**	6	6	6	=	μ({α, β, γ})
		2	$\underline{3}$	6	8	6	**8**	6	6	=	μ({α, β, δ})
		2	4	$\underline{5}$	8	6	6	**8**	6	=	μ({α, γ, δ})
		2	4	6	$\underline{7}$	6	6	6	**8**	=	μ({β, γ, δ})
		$\underline{1}$	$\underline{3}$	6	8	**8**	**8**	6	6	=	μ({α, β})
		$\underline{1}$	4	$\underline{5}$	8	**9**	6	**8**	6	=	μ({α, γ})
		$\underline{1}$	4	6	$\underline{7}$	**8**	6	6	**8**	=	μ({β, γ})
		2	$\underline{3}$	6	$\underline{7}$	6	**8**	6	**8**	=	μ({β, δ})
		2	4	$\underline{5}$	$\underline{7}$	6	6	**8**	**9**	=	μ({γ, δ})
		2	$\underline{3}$	$\underline{5}$	8	6	**9**	**9**	6	=	μ({α, δ})

5.3 Some Interesting Cases of T(A)

Here we consider two possible arrangements of T(A), the first guarantees necessity of all minima in the tree (i.e. corresponding to the relevant subsets in the tree), while the second maximizes coordinate sharing among the profiles. As we will see, both cases require less minima than the worst case scenario examined above.

T(A) with only necessary minima: When the daughters of any set in T(A) partition the set, then every node in T(A) yields a necessary μ. In this case, every set of profiles in the tree will have the shape shown below, with no profile ever hosting two minimal coordinate values across the set. When building the next layer of T(A), only the profiles sharing a minimal value *m* will be selected, with empty intersection between the various subsets.

(39) $S=\{ \langle m_1, V_2, ..., V_n\rangle, \quad \langle V_1, m_2, ..., V_n\rangle, ... , \quad \langle V_1, V_2, ..., m_n\rangle$
$\langle m_1, V'_2, ..., V'_n\rangle, \quad \langle V'_1, m_2, ..., V'_n\rangle, ... , \quad \langle V'_1, V'_2,..., m_n\rangle$
$.......... \}$

The minimum for the entire set is $\mu^S=\langle m_1+1, m_2+1, ..., m_n+1\rangle$, and it cannot bound the μ of each subset *i(S)*, which would post the lower value m_i on the i^{th} coordinate. Likewise, each subset minimum cannot bound μ^S, since their value on some other coordinate is higher than $m_i+1=\mu^S(i)$. This is ensured by the empty intersection that must hold across the subsets. For a subset to match the value of μ^S on all coordinates other than *i*, and thus bound it via its own minimum, some of its profiles would have to contain a minimal value on some coordinate other than *i*. For example, a set collecting the profiles with minimal i^{th} coordinate with the profile $p=\langle ..., m_i, ..., m_j, ...\rangle$ would make it possible for the corresponding μ to match μ^S on the j^{th} coordinate. But *p* would of course now be part of two subsets, namely *i(S)* and *j(S)*, violating the intersection requirement.

For T(A) to only allow necessary minima, the condition must hold on every node, and hence the schema shown for S in the above example must reoccur on each of S relevant subsets, where it would affect the V-values shown above, which thus should not be considered to be unconstrained. The practical example below shows how a complete case could look like.

A final problem arises with singleton subsets, where μ's coincide with profiles in A. To avoid bounding, each profile must show at least one value which is 2 violations higher than any minimal coordinate of the subsets dominating it in T(A). The example below conforms to all conditions examined here. The next one is similar, but lets the singleton sets bounding the subsets immediately dominating them.

(40) $S=\{ \langle 0, 1, 3\rangle, \langle 1, 0, 3\rangle, \langle 1, 3, 0\rangle,$ $\quad \mu^S=\langle 1,1,1\rangle$
$\langle 0, 3, 1\rangle, \langle 3, 0, 1\rangle, \langle 3, 1, 0\rangle \}$

$1(S)= \{\langle 0, 1, 3\rangle, \langle 0, 3, 1\rangle\}$ $\quad \mu^{1(S)}=\langle 0,2,2\rangle$
$2(S)= \{\langle 1, 0, 3\rangle, \langle 3, 0, 1\rangle\}$ $\quad \mu^{2(S)}=\langle 2,0,2\rangle$
$3(S)= \{\langle 1, 3, 0\rangle, \langle 3, 1, 0\rangle\}$ $\quad \mu^{3(S)}=\langle 2,2,0\rangle$

$2(1(S)=\{\langle 0, 1, 3\rangle\}$ $\quad \mu^{2(1(S))}=\langle 0, 1, 3\rangle$
$3(1(S)=\{\langle 0, 3, 1\rangle\}$ $\quad \mu^{3(1(S))}=\langle 0, 3, 1\rangle$

$1(2(S)=\{\langle 1, 0, 3\rangle\}$ $\quad \mu^{1(2(S))}=\langle 1, 0, 3\rangle$
$3(2(S)=\{\langle 3, 0, 1\rangle\}$ $\quad \mu^{2(1(S))}=\langle 3, 0, 1\rangle$

$1(3(S)=\{\langle 1, 3, 0\rangle\}$ $\quad \mu^{1(3(S))}=\langle 1, 3, 0\rangle$
$2(3(S)=\{\langle 3, 1, 0\rangle\}$ $\quad \mu^{2(3(S))}=\langle 3, 1, 0\rangle$

(41) $S=\{\ \langle 0, 1, 2\rangle,\ \langle 1, 0, 2\rangle,\ \langle 1, 2, 0\rangle,\ \langle 0, 2, 1\rangle,\ \langle 2, 0, 1\rangle,\ \langle 2, 1, 0\rangle\ \}$ $\qquad \mu^{S}=\langle 1,1,1\rangle$

$1(S)= \{\langle 0, 1, 2\rangle, \langle 0, 2, 1\rangle\}$ $\qquad \mu^{1(S)}=\langle 0,2,2\rangle$
$2(S)= \{\langle 1, 0, 2\rangle, \langle 2, 0, 1\rangle\}$ $\qquad \mu^{2(S)}=\langle 2,0,2\rangle$
$3(S)= \{\langle 1, 2, 0\rangle, \langle 2, 1, 0\rangle\}$ $\qquad \mu^{3(S)}=\langle 2,2,0\rangle$

$2(1(S)=\{\langle 0, 1, 2\rangle\}$ $\qquad \mu^{2(1(S))}=\langle 0, 1, 2\rangle$ bounds $\mu^{1(S)}$
$3(1(S)=\{\langle 0, 2, 1\rangle\}$ $\qquad \mu^{3(1(S))}=\langle 0, 2, 1\rangle$ bounds $\mu^{1(S)}$

$1(2(S)=\{\langle 1, 0, 2\rangle\}$ $\qquad \mu^{1(2(S))}=\langle 1, 0, 2\rangle$ bounds $\mu^{2(S)}$
$3(2(S)=\{\langle 2, 0, 1\rangle\}$ $\qquad \mu^{2(1(S))}=\langle 2, 0, 1\rangle$ bounds $\mu^{2(S)}$

$1(3(S)=\{\langle 1, 2, 0\rangle\}$ $\qquad \mu^{1(3(S))}=\langle 1, 2, 0\rangle$ bounds $\mu^{3(S)}$
$2(3(S)=\{\langle 2, 1, 0\rangle\}$ $\qquad \mu^{2(3(S))}=\langle 2, 1, 0\rangle$ bounds $\mu^{3(S)}$

Note that this class of profile sets cannot give rise to the worst scenario independently of the number of dimensions. The condition on subset-partition goes directly against coordinate sharing, making it impossible, except for K=2, to construct K subsets within T(A) each containing K−1 profiles, since this would inevitably require some profiles to occur in more than one subset.

Maximal Sharing. An interesting case arises when profiles are allowed to maximize coordinate sharing. If we set the number of profiles equal to that of the allowed dimensions, the profile set will look like the schema below, with each two profiles diverging on exactly two coordinates *i* and *j*.

(42)
$$
\begin{array}{llllllll}
A=\{ & \langle & m_1, & m_2, & m_3, & \ldots, & m_{k-1}, V_k & \rangle_1, \\
 & \langle & m_1, & m_2, & m_3, & \ldots, & V_{k-1}, m_k & \rangle_2, \\
 & \langle & m_1, & m_2, & m_3, & \ldots, & m_{k-1}, m_k & \rangle_3, \\
 & & \ldots & & \ldots & & \ldots & \\
 & \langle & m_1, & m_2, & V_3, & \ldots, & m_{k-1}, m_k & \rangle_{k-2}, \\
 & \langle & m_1, & V_2, & m_3, & \ldots, & m_{k-1}, m_k & \rangle_{k-1}, \\
 & \langle & V_1, & m_2, & m_3, & \ldots, & m_{k-1}, m_k & \rangle_k \quad \}
\end{array}
$$

Provided that each V_i is at least 2 units greater then the corresponding minimum, each of the $\binom{K}{2}$ pairs of profiles will yield a necessary μ, because it will beat each candidate on coordinates *i* and *j*. On the other hand, the minima of any larger set will be bounded by those of its relevant subsets, because the subsets will share across all their profiles some additional minimal value m_i not shared by the superset. The μ for the superset will thus post a value m_i+1 against the lower m_i value of the subset minimum, while on any other coordinate superset and subset will be identical. The overall number of necessary minima will thus be K+$\binom{K}{2}$, hence equal to $\frac{K(K+1)}{2}$. An example follows below.

(43) S={ ⟨0, 0, 2⟩, ⟨0, 2, 0⟩, ⟨2, 0 0⟩ } μ^S=⟨1,1,1⟩

1(S)= {⟨0, 0, 2⟩, ⟨0, 2, 0⟩}	$\mu^{1(S)}$=⟨0,1,1⟩	bounds μ^S
2(S)= {⟨0, 0, 2⟩, ⟨2, 0 0⟩}	$\mu^{2(S)}$=⟨1,0,1⟩	bounds μ^S
3(S)= {⟨0, 2, 0⟩, ⟨2, 0 0⟩}	$\mu^{3(S)}$=⟨1,1,0⟩	bounds μ^S
2(1(S)=1(2(S)={⟨0, 0, 2⟩}	μ=⟨0, 0, 2⟩	
3(1(S)=1(3(S)={⟨0, 2, 0⟩}	μ=⟨0, 2, 0⟩	
3(2(S)=2(3(S)={⟨2, 0 0⟩}	μ=⟨2, 0 0⟩	

Appendix

The Defeating-Bounding lemma and the Defeating theorem are repeated below together with their demonstration. For an intuitive outline, see p. 6.

(44) **Lemma. Defeating-Bounding.** Let A be a profile set and λ a profile in V. Then A is a non-empty defeating set for λ if and only if there exists a non-empty set B in A that constitutes a bounding set for λ.

$$\forall A \neq \emptyset,\ \lambda \in D(A) \Leftrightarrow \exists B \subseteq A,\ B \neq \emptyset,\ B = B(\lambda).$$

Pf. ($\Leftarrow$) By hypothesis, $B \neq \emptyset$ is a bounding set for λ, hence it satisfies reciprocity, therefore B is also a non-empty defeating set for λ.

($\Rightarrow$)

1. Let $A \neq \emptyset$ be a defeating set for λ. Then A satisfies reciprocity with respect to λ. Moreover, by definition of defeating set, $\lambda \notin A$.
2. Let B be the subset of A formed by collecting each α in A beating λ on some coordinate, i.e. $B = \{\alpha: \alpha \in A \text{ and } \exists i,\ \alpha(i) < \lambda(i)\}$.
3. B satisfies strictness by definition.
4. B satisfies reciprocity; let *i* be a coordinate such that $\lambda(i) < \beta(i)$ for some $\beta \in B$. Then by reciprocity on A $\exists \alpha,\ \alpha(i) < \lambda(i)$. By definition of B, $\alpha \in B$.
5. Moreover $B \neq \emptyset$, because for any $\alpha \in A$, given $\lambda \notin A$ from 1 above, it follows $\lambda \neq \alpha$, and therefore $\exists i,\ \lambda(i) \neq \alpha(i)$. If $\lambda(i) > \alpha(i)$, then $\alpha \in B$, else by reciprocity $\exists \alpha',\ \alpha'(i) < \lambda(i)$.
6. By 3, 4, and 5, set B is a non-empty bounding set for λ. □

The defeating and bounding lemma extends to defeating sets all the properties of bounding sets associated with the bounding theorem, namely that if A is a non-empty defeating set for λ, then λ is a loser in any optimization involving A. The corresponding defeating theorem is demonstrated below.

(45) **Defeating Theorem.** Let Σ be a set of constraint coordinates for V, and let K be a profile set in V and λ a profile in K. Then λ is suboptimal in K under any ranking of Σ iff there is in K a non-empty defeating set A for λ.

$$\lambda \notin W(K, \Sigma) \Leftrightarrow \exists A \neq \emptyset,\ A \subseteq K,\ A \sqsubset \lambda.$$

Pf. ($\Leftarrow$) By the defeating bounding lemma, set A contains a non-empty bounding subset A′ for λ. By the bounding theorem, λ is a loser on any optimization involving A′, and hence in K as well.

($\Rightarrow$) Let λ be a loser in K. Then, by the bounding theorem, there is in K a non-empty bounding set A in V satisfying strictness and reciprocity with respect to λ. Since it satisfies reciprocity, A also qualifies as defeating set for λ in V. □

As already briefly explained earlier on, despite their close relation, bounding and defeating sets are not identical. Consider the following informative example. Let $A = \{\langle 0,4 \rangle_\alpha, \langle 4,0 \rangle_\beta, \langle 2,2 \rangle_\gamma)$.

Set A qualifies as bounding set for $\lambda_1=\langle 1,3\rangle$, with α and γ satisfying strictness on i=1 and β satisfying it on i=2. As the reader may check, A also satisfies reciprocity. Since bounding sets are also defeating sets, A also qualifies as defeating set for λ_1.

Now consider $\lambda_2=\langle 1,1\rangle$. Suddenly, γ is no longer able to satisfy strictness. As a consequence, A can no longer qualify as bounding set for λ_2. It however still qualifies as defeating set for λ_2, because reciprocity remain satisfied, with α acting as a reciprocity rescuer on i=1 and β on i=2.

The reason for this asymmetry is that γ is itself collectively bounded by B=$\{\alpha, \beta\}$, but ruins A's prospects to qualify as bounding set for λ_2 due to its failure of strictness. Of course, λ_2 is the minimum of A, i.e. $\mu^A= \langle 1,1\rangle=\lambda_2$.

The notion of defeating set permits us to hold true for any set A that the set of profiles defeated by the set minimum includes all profiles defeated by A itself (with the exception of the minimum), i.e. that $D(A)=D(\mu^A)+\{\mu^A\}$. The notion of bounding set does not allow for the same straightforward expression, because of problematic cases like the one just discussed arise whenever A contains profiles bounded by some of its subsets. If we temporarily interpret D(A) as meaning 'profiles bounded by the bounding set A' the above equality would be falsified by the example just considered, since μ^A is not bounded by A=$\{\alpha, \beta, \gamma\}$ due to the failure of strictness by γ. Nor does the problem only concerns minima like μ^A: the profiles $\langle 1,2\rangle$ and $\langle 2,1\rangle$ are not minimal but they too fail to be bounded by A due to a strictness failure. The equation does hold of bounding sets when these contain only winners, but this assumption would diminish the generality of our results, besides creating further difficulties when calculating L(A).

References

ROA = Rutgers Optimality Archive, http://ruccs.rutgers.edu/ROA.html
RuCCS-TR = Technical Reports of the Rutgers Center for Cognitive Science.

Golston, Chris. 1996. Direct Optimality Theory: Representation as pure markedness. *Language* 72.4, 713-748. ROA-71.

Grimshaw, Jane. 1997. Projection, Heads, and Optimality. *Linguistic Inquiry* 28, 373-422. ROA-68.

Prince, Alan, and Paul Smolensky. 1993. *Optimality Theory: Constraint Interaction in Generative Grammar*. RuCCS-TR-2.

Samek-Lodovici, Vieri. 1992. Universal Constraints and Morphological Gemination: Crosslinguistic Study. ROA-149.

Samek-Lodovici Vieri, and Alan Prince. 1999. Optima. RuCCS-TR-57. ROA-363.

Samek-Lodovici Vieri, and Alan Prince. 2000. Finite Generation. Handout of talk at the Rutgers Optimality Research Group. Rutgers University.

Tesar, Bruce. 1995. Computational Optimality Theory. PhD Dissertation, University of Colorado at Boulder. ROA-90.

Tesar, Bruce. 2000. Using Inconsistency Detection to Overcome Structural Ambiguity in Language Learning. RuCCS-TR-58. ROA-426.

Tesar, Bruce, and Paul Smolensky. 1998. Learnability in Optimality Theory. *Linguistic Inquiry* 29.2, 229-268. ROA-155, ROA-156.

Tesar, Bruce, and Paul Smolensky. 2000. *Learnability in Optimality Theory*. MIT Press: Cambridge, MA.